TOTALLY ORTHOGONAL COMPLEMENTARY BINARY CODED SEQUENCES AND APPLICATIONS TO COMMUNICATIONS SYSTEMS

Charles Everett French

United States
Naval Postgraduate School

THESIS

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by

Charles Everett French

Thesis Advisor: S. Jauregui, Jr.

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Charles Everett French
Lieutenant, United States Navy
B.S., United States Naval Academy, 1963

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ABSTRACT

Complementary binary code sequences were invented by M. J. E. Golay in the investigation of infrared multislit spectrometry. Complementary coding sequences have the property of an infinite correlation peak to peak ambiguity ratio when detected with a matched filter.

Cooperative or totally orthogonal complementary code pairs are two sets of complementary pairs such that the cross correlation is zero in every position. A proof is given that every complementary pair has two totally orthogonal pairs, i.e., one the complement of the other. A proof that these pairs are the only pairs is also given.

A communication system involving complementary code binary sequences is simulated on the hybrid computer and compared with an ideal receiver for an uncoded signal. By using both the totally orthogonal code and time shifting, a method of horizontal multiplexing of binary coded information is proposed and evaluated. Various other complementary coding systems and possible uses are discussed.

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Thanks are extended to the W. R. Church Computer Facility for the computer time for searching for various totally orthogonal sets of codes and correlation of various length codes. Thanks are also extended for the computer facilities in Spanagel Hall, particularly the hybrid computer consisting of the XDS-9300 and the CI-5000, which were used extensively in simulation of multiplexed communication systems using complementary sequences.

I. INTRODUCTION

As defined by M. J. E. Golay [Ref. 1], a set of complementary sequences is a pair of equally long, finite sequences of two kinds of elements which have the property that the number of like elements with any given separation in one series is equal to the number of pairs of unlike elements with the same separation in the other series for all separations.

Example:

A. 0 0 0 1 0 0 1 0

B. 0 0 0 1 1 1 0 1

Consider series A at a separation of one element, counting the number of likes:

```
0 0 0 1 0 0 1 0
 l l u u l u u        3 likes
```

Next, counting the number of unlikes in series B with a separation of one element:

```
0 0 0 1 1 1 0 1
 l l u l l u u        3 unlikes
```

Similarly for longer separations the number of likes in A and the number of unlikes in B are listed below:

Separation Distance	Number of Likes A	Number of Unlikes B
2	3	3
3	4	4
4	2	2
5	2	2
6	1	1
7	1	1

The basic property is also expressed in autocorrelative terms. Let the a_i and b_i elements ($i=1,2,\cdots,n$) of two n-long complementary sequences be either +1 or -1, and let the respective autocorrelative series be described by the series c_j and d_j respectively, with subscript ranging over the integers from $-n+1$ to $n-1$, and defined by the respective equations.

$$\left.\begin{aligned} c_j &= \sum_{i=1}^{i=n-j} a_i a_{i+j} \\ d_j &= \sum_{i=1}^{i=n-j} b_i b_{i+j} \end{aligned}\right\} \quad \text{for } j \geq 0$$

$$\left.\begin{aligned} c_j &= c_{-j} \\ d_j &= d_{-j} \end{aligned}\right\} \quad \text{for } j < 0$$

Then:

$$c_j + d_j = 0 \qquad \text{for } j \neq 0$$

$$c_o + d_o = 2n$$

The first use of these complementary sequences was by Golay in the field of multislit spectrometry. Since then, they have been proposed for radar and sonar. Golay mentioned in 1961 that these sequences might be applied to communication systems [Ref. 1].

One problem in radar and sonar systems is to get a maximum amount of energy radiated for long-range detection, but at the same time keep good range resolution. Three basic methods are available: FM modulated wave, integration of successive pulses, and intrapulse coding. Of these only intrapulse coding will be discussed.

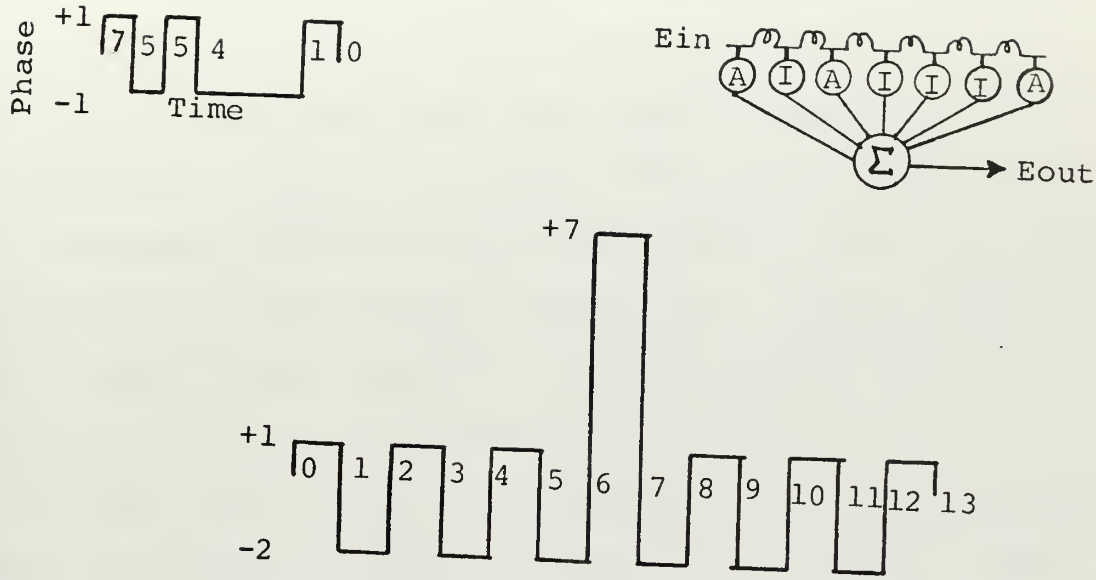


Figure 1. Matched filter detection of a random code.

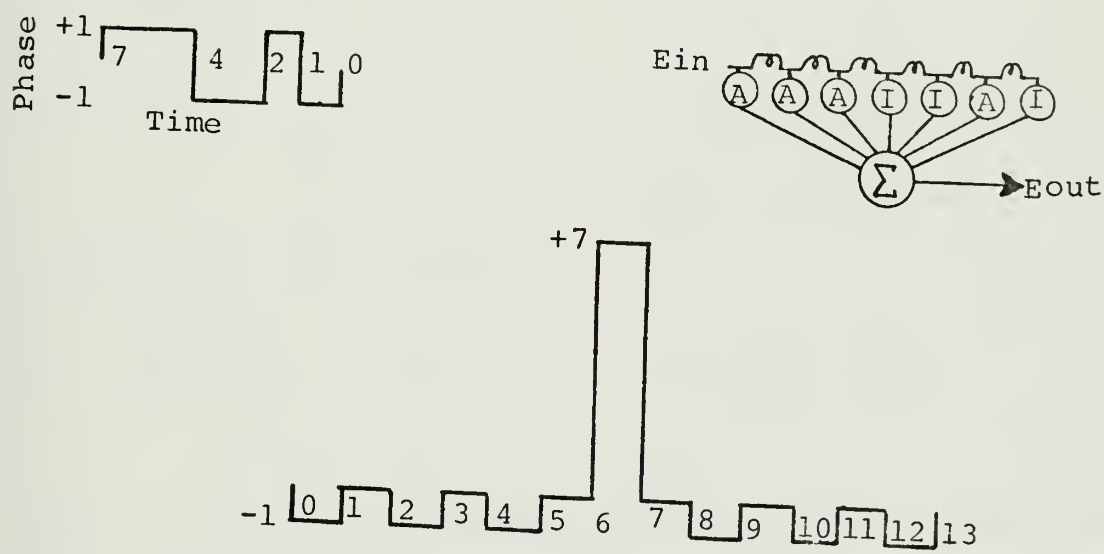


Figure 2. Matched filter detection of an optimal noise code.

Using intrapulse coding with a matched filter is illustrated in Fig. 1. Note that there is a certain amount of coded signal output from the filter which cannot be completely eliminated. However, by proper choice of the coded pulse with its matched filter it can be minimized. See Fig. 2 for an illustration of an "ideal" code filter. It is possible to use two complementary codes to eliminate clutter completely. An example of this is illustrated in Fig. 3.

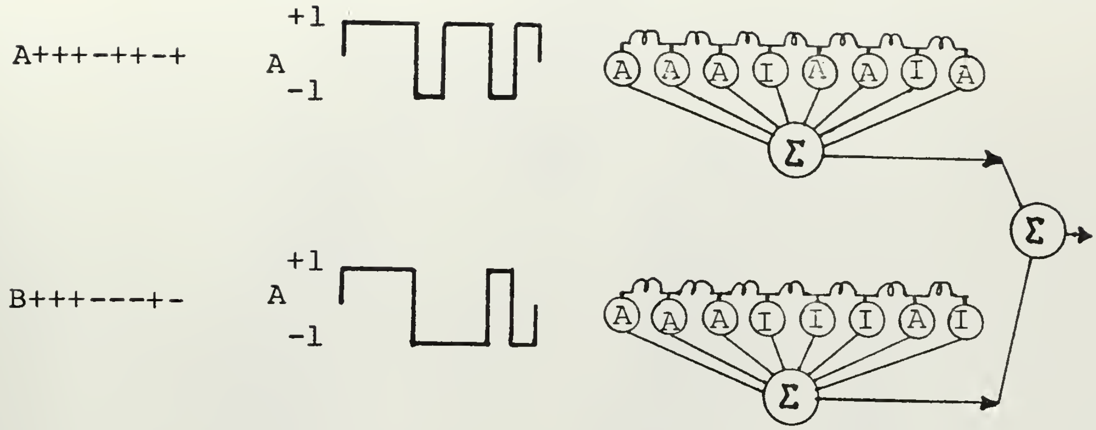
The basic way to make a complementary pair communication system is to have two symbols representing a 1 and -1, and send either one code or the complement of the code and to receive the signal in a matched filter. See Fig. 4.

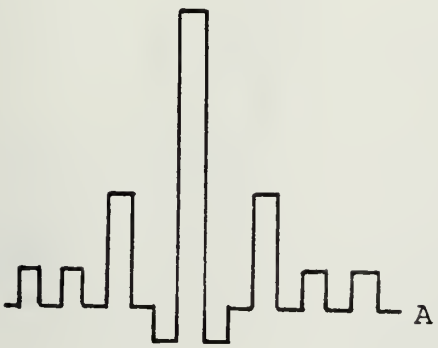
Example:

	Signal for +1	Filter	Correlation
Ch 1	1 1 1-1	1 1 1-1	-1 0 1 4 1 0-1
Ch 2	1 1-1 1	1 1-1 1	1 0-1 4-1 0 1
			0 0 0 8 0 0 0
			Signal decoded as +1

	Signal for -1	Filter	
Ch 1	-1-1-1+1	1 1 1-1	1 0-1-4-1 0 1
Ch 2	-1-1+1-1	1 1-1 1	-1 0 1-4 1 0-1
			0 0 0-8 0 0 0
			Signal decoded as -1

The property that eliminates clutter from the radar system and allows the improvement in resolution from a series of pulses is the orthogonality of the code with a time shift of itself. In a communication system this would allow the sender to start the next message one bit behind the previous one and by superposition of the linear system it can be seen that these bits will be separated at the receiver end. In the





A	+1	0	+1	0	+3	0	-1	+8	-1	0	+3	0	+1	0	+1	0
+																
B	-1	0	-1	0	-3	0	+1	+8	+1	0	-3	0	-1	0	-1	0
Sum	0	0	0	0	0	0	0	16	0	0	0	0	0	0	0	0

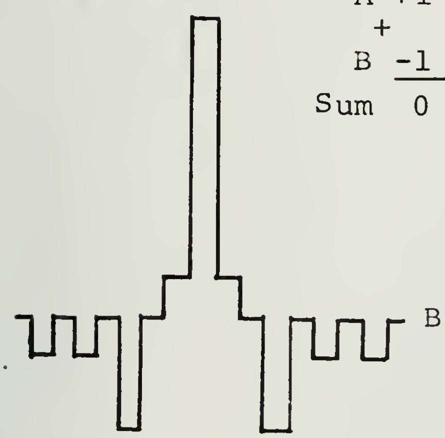


Figure 3. Matched Filter Detection of a Complementary Code of Length 8.

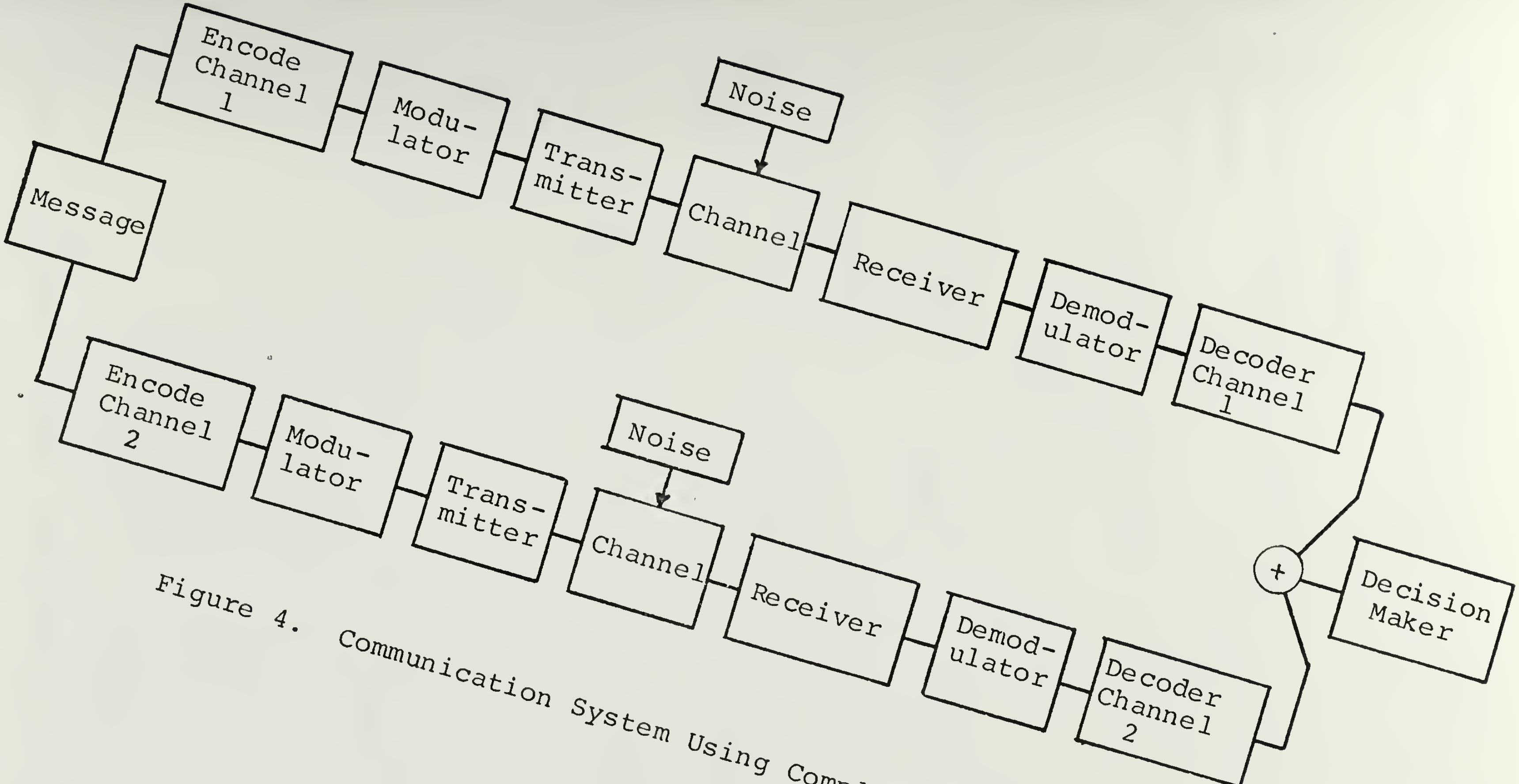


Figure 4. Communication System Using Complementary Code

example below a message is sent with the first bit on the right, and in the correlation the first bit is on the right:

Example:

```
Encoding Channel 1    Code 111-1

Message
-111-1-1
        -1-1-1+1    first bit
      -1-1-1+1    second
    +1+1+1-1    etc.
  +1+1+1-1          Channel 1 filter       Correlation
-1-1-1+1
-1 0+1+2-2-3 0+1      +1+1+1-1      +1-1-2-2+5+4-3-6-2+1+1


Encoding Channel 2
     Code 11-11          filter
Message                  +1+1-1+1      -1+1+2-6+3+4-5-2+2-1-1
-111-1-1                                0 0 0-8+8+8-8-8 0 0 0
        -1-1+1-1
      -1-1+1-1                             Message Out
    +1+1-1+1
  +1+1-1+1
-1-1+1-1
-1 0+3-2-2+1 0-1
```

This could be thought of as five messages added together:

First Message	0 0 0 0-1
Second Message	0 0 0-1 0
Third Message	0 0+1 0 0
Fourth Message	0+1 0 0 0
Fifth Message	-1 0 0 0 0
Sum	-1+1+1-1-1

Using superposition the same results are obtained

From message	1	0 0 0 0 0 0 0 −8 0 0 0
	2	0 0 0 0 0 0 −8 0 0 0 0
	3	0 0 0 0 0 +8 0 0 0 0 0
	4	0 0 0 0 +8 0 0 0 0 0 0
	5	0 0 0 −8 0 0 0 0 0 0 0
		0 0 0 −8 +8 +8 −8 −8 0 0 0

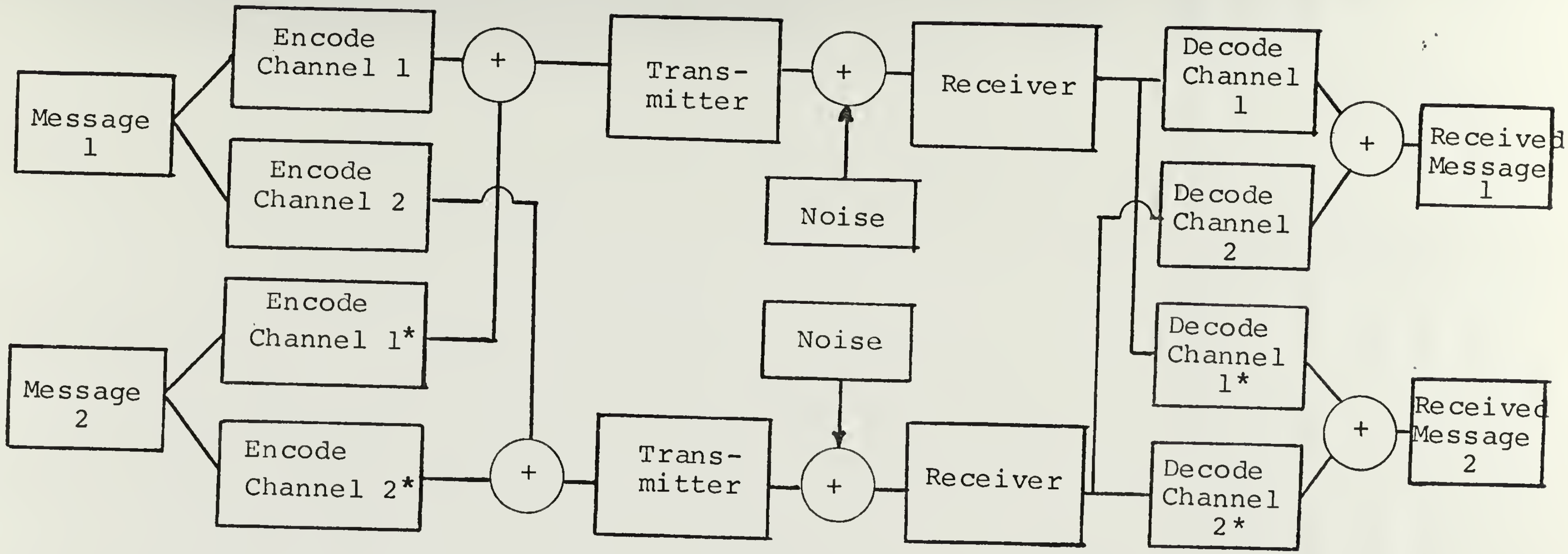
Two complementary codes are defined to be totally orthogonal if their cross correlation is zero in every position. Although it was previously found in Ref. 3 that these totally orthogonal codes could be generated from shorter codes for for lengths which were a power of two, in this thesis the author will show that for every complementary code there are two codes that are totally orthogonal to the original code. An example of two totally orthogonal codes is shown below.

Example:

	Original Code	Totally Orthogonal Code	Cross Correlation
Channel 1	+1+1+1−1	−1+1−1−1	−1−2−1 0+1−2+1
Channel 2	+1+1−1+1	−1+1+1+1	+1+2+1 0−1+2−1
			0 0 0 0 0 0 0

Since the totally orthogonal code is also a complementary pair, messages can be encoded both on the original code and on the orthogonal code and its complement. The results can be added algebraically and transmitted. Thus the information on a pair of bandwidths is doubled (See Fig. 5).

By using the results of sending messages delayed only one bit and by also using the orthogonal complementary pair, it is possible to send two signals per bit rate on the two



* Totally Orthogonal Code

Figure 5. Communication System Using Totally Orthogonal Complementary Pair Sequences.

channels. If the code length is n, then each bit on each channel would have components from 2n messages.

A detailed simulation of the above horizontal multiplexing was done on the XDS-9300 and CI-5000 hybrid computer, and comparison made with an ideal uncoded system. A number of variations of communication systems are proposed and discussed based on complementary binary sequences.

II. COMPLEMENTARY SEQUENCES

The purpose of this chapter is to provide the reader with a background on complementary pair sequence codes. The first part of the chapter is from Golay's "Complementary Series." No attempt is made to prove the theorems. Proofs are available from either Golay [Ref. 1] or Jauregui [Ref. 2]. In the second part of the chapter totally orthogonal codes are covered and the proofs appearing there are the work of the author.

A. GENERAL PROPERTIES

Complementary pairs may be represented by either ones and zeros using modulo two addition, or by plus and minus ones using multiplication. The convention of plus and minus ones will ordinarily be used since it is usually more convenient. However, some properties of complementary pairs from time to time can be more conveniently represented by ones and zeros. Much of the literature on complementary pairs is expressed in ones and zeros and modulo two addition.

1. Transformations

A single pair of complementary series can be the basis for the construction of 64 pairs of complementary series (some of which may be identical in shorter code lengths) by either performing or not performing the following six operations:

a. Interchanging the series.

b. Reversing the first series.

c. Reversing the second series.

d. Complementing the first series.

e. Complementing the second series.

f. Complementing the elements of even order of each series. Complementation is changing to the other symbol. If the symbols are ones and zeros the ones become zeros and the zeros ones. If the symbols are plus and minus ones, the signals are reversed.

2. <u>Possible Code Lengths</u>

The code length must be even, and must also be the sum of, at most, two squares. From these two restrictions the allowable lengths up to 50 are:

2, 4, 8, 10, 16, 18, 20, 26, 32, 34, 36, 40, and 50.

3. <u>General Synthesis</u>

Given a complementary pair A and B, a new complementary pair of twice that length can be formed by AB and $A\overline{B}$ where $\overline{B}$ is the complement of B. For example:

Given the complementary pair:

A = 1 1 1-1

B = 1 1-1 1

The new complementary pair formed by this method is:

AB = 1 1 1-1 1 1-1 1

$A\overline{B}$ = 1 1 1-1-1-1 1-1

In addition, a few other methods of producing new codes of double the original code length are also available.

4. <u>Kernels</u>

Since methods are available for generating codes twice a known code length, the problem of finding codes of

different lengths is principally one of finding kernels of the shortest length that can be combined into the desired length. Kernels have been found for lengths 2, 10, and 26 by Golay [Ref. 1]. Jauregui [Ref. 2] verified by an exhaustive computer search that the kernel found for length 26 and its transformations were the only codes to exist at that length. Summarized below is the current status of the search for kernels.

Code Length	Number of Kernels (excluding transformations)
2	1
10	2
18	None exist [Ref. 1]
26	1
34	None found after extensive but not exhaustive search [Ref. 2].
50	It has been hypothesized that none exist [Ref. 2].

B. TOTALLY ORTHOGONAL COMPLEMENTARY SEQUENCES

Two pairs of complementary sequences A_1, B_1, A_2, B_2 are defined to be "totally orthogonal" if A_1 correlated with A_2 plus B_1 correlated with B_2 is zero. More simply stated two complementary pair sequences are totally orthogonal if the complementary cross correlation is the null vector. Speiser and Whitehouse found a way to generate arbitrarily long "cooperative" or totally orthogonal codes for lengths a power of 2 [Ref. 3].

It will be shown that every complementary code has exactly a set of two codes that are totally orthogonal to it. These totally orthogonal codes are shown to be transformations of

the original complementary code and thus can easily be found for all complementary codes for which kernels are known or that can be built from other codes.

Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$. Then an alternate way of expressing correlation is the sum of the terms in the diagonals of a matrix. The transpose of A is a column matrix and has n rows and one column. B is a row matrix with one row and n columns. The matrix product of A^TB is a n-row, n-column matrix.

Example:

$n=4$

$$A^TB = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} (b_1 \; b_2 \; b_3 \; b_4) = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix}$$

The main diagonal of the above product matrix contains the four terms a_1b_1, a_2b_2, a_3b_3, and a_4b_4. If we number the diagonals of the matrix starting from the upper right-hand corner in this example, the first diagonal contains only the one term a_1b_4. The second diagonal contains the two terms a_1b_3 and a_2b_4. The third diagonal contains three terms; the fourth is the main diagonal. The seventh and last diagonal contains the single term a_4b_1.

Let C be the row matrix $(c_1, c_2, \ldots, c_{2n-1})$ and each term be equal to the sum of the terms in corresponding diagonals of the matrix A^TB. Then:

$$c_1 = a_1 b_n$$
$$c_2 = a_1 b_{n-1} + a_2 b_n$$
$$\vdots$$
$$c_n = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$
$$\vdots$$
$$c_{2n-1} = a_n b_1$$

From term by term observation this is recognized to be the result of putting a signal series specified by A into a filter matched to B. C is the cross correlation of A and B and can be denoted as follows:

$$C = A \otimes B = \text{diag } (A^T B)$$

The symbol x is used to mean correlation. The abbreviation diag is used to denote the row matrix with the number of terms equal to the number of diagonals of the matrix on which it is operating. Each term is equal to the sum of the terms in the corresponding diagonal. From the previous definition of autocorrelation, the same numbers result, but the subscripts in the case of autocorrelation range from -n+1 to n-1. In the case of autocorrelation this presented no difficulty since the terms with negative subscripts were equal to the terms with positive subscripts. However in the general case it is more convenient to allow only positive subscripts, particularly since FORTRAN only allows positive subscripts.

1. <u>Theorems Concerning Totally Orthogonal Codes</u>

In this section the author will develop the theorem showing that the totally orthogonal codes are transformations of the original codes. In the development other theorems

regarding linearity, effects of changing the order of correlation, and effects of reversing the codes themselves will be proven.

a. Complementary Correlation

Let A and B be the two sequences of a complementary pair, and C and D be another complementary pair. The complementary correlation of the first complementary pair with the second is defined to be A correlated with B plus C correlated with D. This is denoted as:

$$\binom{A}{B} \otimes_c \binom{C}{D} = (A \otimes C) + (B \otimes D)$$

The symbol $\otimes_c$ is used to indicate complementary correlation.

Theorem: Complementary correlation is linear.

That is:

$$\left[\alpha\binom{A}{B} + \beta\binom{C}{D}\right] \otimes_c \binom{E}{F} = \alpha\binom{A}{B} \otimes_c \binom{E}{F} + \beta\binom{C}{D} \otimes_c \binom{E}{F}$$

Where A, B, C, D, E, and F are in the form $(a_1, a_2, \cdots, a_n)$, and α and β are real constants. Complementary correlation is the first sequence of the first pair correlated to the first sequence of the second pair, plus the second sequence of the first pair correlated to the second sequence of the second pair. Thus by the distributive properties of matrices and the linearity of the diag operator, the following equations can be written:

$$\begin{aligned} (\alpha A + \beta C) \otimes E &= \mathrm{diag}(\alpha A + \beta C)^T E \\ &= \mathrm{diag}(\alpha A^T E) + \mathrm{diag}(\beta C^T E) \\ &= \alpha\ \mathrm{diag}(A^T E) + \beta\ \mathrm{diag}(C^T E) \end{aligned}$$

Similarly:

$$(\alpha B + \beta D) \otimes F = \alpha\ \mathrm{diag}(B^T F) + \beta\ \mathrm{diag}(D^T F)$$

By adding the above two equations, regrouping, and recognizing that $\alpha\ \mathrm{diag}(A^T E) + \alpha\ \mathrm{diag}(B^T F)$ is equal to the complementary correlation of α times the pair A, B with the pair E, F; and that $\beta\ \mathrm{diag}(C^T E) + \beta\ \mathrm{diag}(D^T F)$ is equal to the complementary correlation of β times the pair C, D, with the pair E, F; the following equation is obtained:

$$[\alpha\binom{A}{B} + \beta\binom{C}{D}] \otimes_C \binom{E}{F} = [\alpha\binom{A}{B} \otimes_C \binom{E}{F}] + [\beta\binom{C}{D} \otimes_C \binom{E}{F}]$$

Q. E. D.

b. Theorem: Changing the order of correlation (commutating) results in reversing the correlation.

To take the transpose of a product of matrices, the transpose of each term is taken in reverse order. Thus:

$$(B^T \cdot A)^T = (A^T \cdot B)$$

The order of the diagonals in the transpose of a matrix is reversed.

$$A \otimes B = \mathrm{diag}(A^T \cdot B)$$

$$B \otimes A = \mathrm{diag}(B^T \cdot A) = \mathrm{diag}(A^T \cdot B)^T$$

Since the order of the diagonals is reversed, this is the same as reversing the cross correlation.

Q. E. D.

c. Theorem:, Reversing both codes results in reversing the correlation.

Let $C = A \otimes B$

where $A = (a_1, a_2, \cdots, a_i, \cdots, a_n)$

and $B = (b_1, b_2, \cdots, b_j, \cdots, b_n)$

$C = (c_1, c_2, \cdots, c_{2n-1})$

When correlated the $a_1 b_n$ contributes to the first term, and any increase in the a subscript or decrease in the b subscript increases the subscript on c. Thus $a_i b_n$ contributes to c_i and $a_1 b_j$ contributes to c_{n+1-j} . Then $a_i b_j$ contributes to c_{n+i-j} . Reversing A and B results the following sequences:

$\underline{A} = (a_n, a_{n-1}, \cdots, a_i, \cdots, a_1)$

$\underline{B} = (b_n, b_{n-1}, \cdots, b_j, \cdots, b_1)$

The i_{th} term in the A sequence is the $(n+1-i)_{th}$ term in the $\underline{A}$ sequence, and the j_{th} term in the B sequence becomes the $(n+1-j)_{th}$ term in the $\underline{B}$ sequence. These terms contribute to the correlation terms with the subscript n-i+j. Thus the contribution is to the diagonal the same number of terms on the other side of the main diagonal corresponding to a reverse in the order of terms of the correlation. Since this applies to both the first sequences and second sequences of complementary correlation, it also follows for complementary correlation.

Q. E. D.

Corollary: Both reversing the correlation and changing the order results in the same correlation. This is a direct result of applying the preceding two theorems together.

d. Theorem: A totally orthogonal code can always be obtained by reversing, exchanging codes, and complementing either one.

$$\text{Given:} \quad A \otimes B = \mathrm{diag}(A^T \cdot B)$$

then by the corollary:

$$\underline{B} \otimes \underline{A} = \mathrm{diag}(A^T \cdot B)$$

Since the codes are linear:

$$\underline{\overline{B}} \otimes \underline{A} = \underline{B} \otimes \underline{\overline{A}} = -\mathrm{diag}(A^T \cdot B)$$

Thus: $(A \otimes B) + (\underline{\overline{B}} \otimes \underline{A}) = 0$ for every term.

And also $(A \otimes B) + (\underline{B} \otimes \underline{\overline{A}}) = 0$ for every term.

It should be noted that the above result is not dependent upon A and B being a complementary pair. The cross correlation will be zero as long as both sequences are the same length. Even if each term in the sequence is real or even complex it will be true.

2. Examples

A few examples are given below. Finding the orthogonal codes of a given code of length four is done in a step by step procedure to demonstrate the techniques. The more general case of finding the totally orthogonal complementary pair sequence pair of length n is done and the complementary correlation is shown to be zero.

All complementary codes of length four are grouped into eight groups of a code and its complement and the two codes totally orthogonal to them. For any general code there are 64 transformations. These transformations are shown to partition the 64 transformations into 16 groups, each

containing a code, its complement, and the two codes totally orthogonal to them.

a. Complementary Pair of Length Four

As the first example, a complementary pair sequence of length four whose first sequence is A = +1 +1 +1 -1 and whose second sequence is B = +1 +1 -1 +1 is considered. To get the totally orthogonal sequences using the preceding theorem, this pair must be reversed, exchanged and either sequence complemented. Underlining is used to denote the reverse; while a line over the symbol denotes the complement.

Reversing:

$$\underline{A} = -1 \; +1 \; +1 \; +1$$

$$\underline{B} = +1 \; -1 \; +1 \; +1$$

Exchanging:

$$\underline{B} = +1 \; -1 \; +1 \; +1$$

$$\underline{A} = -1 \; +1 \; +1 \; +1$$

Complementing either one:

$$\overline{\underline{B}} = -1 \; +1 \; -1 \; -1$$

$$\underline{A} = -1 \; +1 \; +1 \; +1$$

Or:

$$\underline{B} = +1 \; -1 \; +1 \; +1$$

$$\overline{\underline{A}} = +1 \; -1 \; -1 \; -1$$

The first one will be demonstrated to be totally orthogonal. That is:

$$\begin{matrix} A \\ B \end{matrix} \otimes_c \begin{matrix} \overline{\underline{B}} \\ \underline{A} \end{matrix} = 0$$

From the definition of complementary correlation and the

representation of correlation presented previously, the following equation can be written:

$$\begin{matrix} A \\ B \end{matrix} \otimes_c \begin{matrix} \overline{B} \\ \underline{A} \end{matrix} = A \otimes \underline{\overline{B}} + B \otimes \underline{A} = \mathrm{diag}(A^T\underline{\overline{B}}) + \mathrm{diag}(B^T\underline{A})$$

$$A^T\underline{\overline{B}} = \begin{bmatrix} +1 \\ +1 \\ +1 \\ -1 \end{bmatrix} (-1\ +1\ -1\ -1) = \begin{bmatrix} -1 & +1 & -1 & -1 \\ -1 & +1 & -1 & -1 \\ -1 & +1 & -1 & -1 \\ +1 & -1 & +1 & +1 \end{bmatrix}$$

$$A \otimes \underline{\overline{B}} = \mathrm{diag}(A^T\underline{\overline{B}}) = (-1\ -2\ -1\ 0\ +1\ -2\ +1)$$

$$B^T\underline{A} = \begin{bmatrix} +1 \\ +1 \\ -1 \\ +1 \end{bmatrix} (-1\ +1\ +1\ +1) = \begin{bmatrix} -1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 \\ +1 & -1 & -1 & -1 \\ -1 & +1 & +1 & +1 \end{bmatrix}$$

$$B \otimes \underline{A} = \mathrm{diag}(B^T\underline{A}) = (+1\ +2\ +1\ 0\ -1\ +2\ -1)$$

Thus: $$\begin{matrix} A \\ B \end{matrix}\ x_c\ \begin{matrix} \overline{B} \\ \underline{A} \end{matrix} = 0$$

Since complementary correlation is a linear operation multiplication by a minus one can be done.

Then:

$$\begin{bmatrix} A \\ B \end{bmatrix} \otimes_c \begin{bmatrix} \overline{B} \\ \underline{A} \end{bmatrix} = 0 \qquad \begin{bmatrix} A \\ B \end{bmatrix} \otimes_c \begin{bmatrix} \underline{B} \\ \underline{\overline{A}} \end{bmatrix} = 0$$

Thus the second is also shown to be totally orthogonal to the original code.

b. Complementary Pair of Length n

As an example the above theorem will be applied to a complementary pair binary coded sequence length n where each term is either a plus or minus one. Let A and B

designate the first and second sequence of the complementary pair.

$$A \otimes A + B \otimes B = (0,\cdots,0,2n,0,\cdots,0)$$

The totally orthogonal codes in this case are $\bar{\underline{B}}$, $\underline{A}$ and $\underline{B}$, $\bar{\underline{A}}$. This first pair of these when written out term by term is:

$$\bar{\underline{B}} = (-b_n,-b_{n-1},\cdots,-b_1)$$

$$\underline{A} = (a_n,a_{n-1},\cdots,a_1)$$

This first pair of these is shown to be totally orthogonal to the original pair, but since complementary correlation is linear to show that the second is totally orthogonal can be very easily done by observing the second pair is just minus one times the first pair as in the previous example.

$$\begin{bmatrix} A \\ B \end{bmatrix} \otimes_c \begin{bmatrix} \bar{\underline{B}} \\ \underline{A} \end{bmatrix} = \operatorname{diag}(A^T\bar{\underline{B}}) + \operatorname{diag}(B^T\underline{A})$$

$$A^T \cdot \bar{\underline{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} (-b_n,-b_{n-1},\cdots,-b_1) = \begin{bmatrix} -a_1b_n & -a_1b_{n-1} & \cdots & -a_1b_1 \\ -a_2b_n & -a_2b_{n-1} & \cdots & -a_2b_1 \\ \vdots & \vdots & & \vdots \\ -a_nb_n & -a_nb_{n-1} & \cdots & -a_nb_1 \end{bmatrix}$$

$$B \times \bar{\underline{A}} = \operatorname{diag}(A^T\ B)$$

$$B^T \cdot A \quad \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} (a_n,\ a_{n-1},\cdots,a_1) = \begin{bmatrix} b_1a_n & b_1a_{n-1} & \cdots & b_1a_1 \\ b_2a_n & b_2a_{n-1} & \cdots & b_2a_1 \\ \vdots & \vdots & & \vdots \\ b_na_n & b_na_{n-1} & \cdots & b_na_1 \end{bmatrix}$$

Thus when the diagonals from above matrices are added together, for every minus number in the upper diagonal the positive term is in the same diagonal in the second matrix. The complementary correlation, which is the sum of the two individual cross correlations, is zero for every term.

c. Codes of Length Four

An exhaustive computer search was done to find all totally orthogonal codes of length four. The thirty-two possible complementary pair sequences including all possible transformations were read into the computer. Only the autocorrelation, the totally orthogonal codes, their correlation, and their complementary correlation were printed out. See the results in Table I.

The results can be summarized by saying for each code and its complement a totally orthogonal code and its complement were found. The thirty-two possible codes including all transformations could be partitioned into eight groups of four. It can be easily verified that each totally orthogonic to another is the reverse, interchange, complement of one.

d. Transformations of Complementary Codes at Any Length.

Previously Golay described 64 transformations that were possible for each complementary code. Jauregui [Ref. 2] showed that they were in fact a closed group, identified all possible operators, and demonstrated an operation multiplication table. Defining operations as in Table II, and using the Jauregui operation multiplication table it can be

TABLE I

EXHAUSTIVE LIST OF TOTALLY ORTHOGONAL COMPLEMENTARY PAIRS OF LENGTH FOUR

Standard Code	Complement Code	Totally Orthogonal Code	Complement Totally Orthogonal Code
+1 -1 -1 -1 -1 +1 -1 -1	-1 +1 +1 +1 +1 -1 +1 +1	+1 +1 -1 +1 -1 -1 -1 +1	-1 -1 +1 -1 +1 +1 +1 -1
+1 -1 -1 -1 -1 -1 +1 -1	-1 +1 +1 +1 +1 +1 -1 +1	+1 -1 +1 +1 -1 -1 -1 +1	-1 +1 -1 -1 +1 +1 +1 -1
-1 -1 -1 +1 -1 +1 -1 -1	+1 +1 +1 -1 +1 -1 +1 +1	+1 +1 -1 +1 +1 -1 -1 -1	-1 -1 +1 -1 -1 +1 +1 +1
-1 -1 -1 +1 -1 -1 +1 -1	+1 +1 +1 -1 +1 +1 -1 +1	+1 -1 +1 +1 +1 -1 -1 -1	-1 +1 -1 -1 -1 +1 +1 +1
-1 +1 -1 -1 +1 -1 -1 -1	+1 -1 +1 +1 -1 +1 +1 +1	+1 +1 +1 -1 -1 -1 +1 -1	-1 -1 -1 +1 +1 +1 -1 +1
-1 -1 +1 -1 +1 -1 -1 -1	+1 +1 -1 +1 -1 +1 +1 +1	+1 +1 +1 -1 -1 +1 -1 -1	-1 -1 -1 +1 +1 -1 +1 +1
-1 +1 -1 -1 -1 -1 -1 +1	+1 -1 +1 +1 +1 +1 +1 -1	-1 +1 +1 +1 -1 -1 -1 -1	+1 -1 -1 -1 +1 +1 -1 +1
-1 -1 +1 -1 -1 -1 -1 +1	+1 +1 -1 +1 +1 +1 +1 -1	-1 +1 +1 +1 -1 +1 -1 -1	+1 -1 -1 -1 +1 -1 +1 +1

Note: Every complementary code of length four appears in the table once. In its row is contained its complement and the two complementary codes that are totally orthogonal to it.

TABLE II

ELEMENTS OF THE UNORDERED OPERATIONS GROUP

Sequence	$A=a_1a_2a_3\cdots a_{n-1}a_n$	$\mathrm{I}=a_1a_3a_5\cdots a_{n-1}$
		$\mathrm{II}=a_na_{n-2}\cdots a_2$
Sequence	$B=b_1b_2b_3\quad b_{n-1}b_n$	$\mathrm{III}=b_1b_3b_5\quad b_{n-1}$
		$\mathrm{IV}=b_nb_{n-2}\quad b_2$

I	=	$(\mathrm{I}\ \mathrm{II}\ \mathrm{III}\ \mathrm{IV})$	R	=	$(\mathrm{I}\ \overline{\mathrm{II}}\ \mathrm{IV}\ \overline{\mathrm{III}})$
A_1	=	$(\overline{\mathrm{I}}\ \mathrm{II}\ \overline{\mathrm{III}}\ \mathrm{IV})$	Q	=	$(\mathrm{II}\ \overline{\mathrm{I}}\ \mathrm{IV}\ \overline{\mathrm{III}})$
A_2	=	$(\mathrm{I}\ \overline{\mathrm{II}}\ \mathrm{III}\ \overline{\mathrm{IV}})$	P	=	$(\overline{\mathrm{II}}\ \mathrm{I}\ \overline{\mathrm{IV}}\ \mathrm{III})$
T_1	=	$(\mathrm{II}\ \mathrm{I}\ \mathrm{III}\ \mathrm{IV})$	O	=	$(\overline{\mathrm{II}}\ \overline{\mathrm{I}}\ \mathrm{III}\ \mathrm{IV})$
T_2	=	$(\mathrm{I}\ \mathrm{II}\ \mathrm{IV}\ \mathrm{III})$	N	=	$(\mathrm{II}\ \mathrm{I}\ \overline{\mathrm{III}}\ \overline{\mathrm{IV}})$
T	=	$(\mathrm{II}\ \mathrm{I}\ \mathrm{IV}\ \mathrm{III})$	M	=	$(\overline{\mathrm{I}}\ \overline{\mathrm{II}}\ \mathrm{IV}\ \mathrm{III})$
C_1	=	$(\mathrm{I}\ \mathrm{II}\ \mathrm{III}\ \mathrm{IV})$	L	=	$(\overline{\mathrm{II}}\ \overline{\mathrm{I}}\ \overline{\mathrm{III}}\ \overline{\mathrm{IV}})$
C_2	=	$(\mathrm{I}\ \mathrm{II}\ \mathrm{III}\ \mathrm{IV})$	K	=	$(\overline{\mathrm{I}}\ \overline{\mathrm{II}}\ \overline{\mathrm{IV}}\ \overline{\mathrm{III}})$
C	=	$(\overline{\mathrm{I}}\ \overline{\mathrm{II}}\ \overline{\mathrm{III}}\ \overline{\mathrm{IV}})$	J	=	$(\overline{\mathrm{II}}\ \overline{\mathrm{I}}\ \overline{\mathrm{IV}}\ \overline{\mathrm{III}})$
Z	=	$(\mathrm{II}\ \overline{\mathrm{I}}\ \overline{\mathrm{III}}\ \mathrm{IV})$	H	=	$(\mathrm{I}\ \mathrm{II}\ \overline{\mathrm{IV}}\ \overline{\mathrm{III}})$
Y	=	$(\overline{\mathrm{II}}\ \mathrm{I}\ \overline{\mathrm{III}}\ \mathrm{IV})$	G	=	$(\overline{\mathrm{II}}\ \overline{\mathrm{I}}\ \mathrm{IV}\ \mathrm{III})$
X	=	$(\overline{\mathrm{II}}\ \mathrm{I}\ \mathrm{III}\ \overline{\mathrm{IV}})$	F	=	$(\mathrm{I}\ \overline{\mathrm{II}}\ \overline{\mathrm{III}}\ \mathrm{IV})$
W	=	$(\mathrm{II}\ \overline{\mathrm{I}}\ \mathrm{III}\ \overline{\mathrm{IV}})$	D	=	$(\overline{\mathrm{I}}\ \mathrm{II}\ \mathrm{III}\ \overline{\mathrm{IV}})$
V	=	$(\overline{\mathrm{I}}\ \mathrm{II}\ \mathrm{IV}\ \overline{\mathrm{III}})$	$\emptyset$	=	$(\mathrm{II}\ \mathrm{I}\ \overline{\mathrm{IV}}\ \overline{\mathrm{III}})$
U	=	$(\overline{\mathrm{I}}\ \mathrm{II}\ \overline{\mathrm{IV}}\ \mathrm{III})$	B	=	$(\overline{\mathrm{II}}\ \mathrm{I}\ \mathrm{IV}\ \overline{\mathrm{III}})$
S	=	$(\mathrm{I}\ \overline{\mathrm{II}}\ \overline{\mathrm{IV}}\ \mathrm{III})$	π	=	$(\mathrm{II}\ \overline{\mathrm{I}}\ \overline{\mathrm{IV}}\ \mathrm{III})$

$$E = (\mathrm{III}\ \mathrm{IV}\ \mathrm{I}\ \mathrm{II})$$

Note: Combining these operations with the E (exchange) operation provides the additional 32 transformations for a total of the 64 possible transformations.

TABLE III

GROUPS OF OPERATIONS PRODUCING TOTALLY ORTHOGONAL PAIRS

Operation	Complement Operation	Totally Orthogonal Operation	Complement Totally Orthogonal Operation	Operations Performed left Hand Member E T_1 C_1 A_2
I	C	EG	EØ	0 0 0 0
A_2	A_1	EB	Eπ	0 0 0 1
C_1	C_2	ET	EJ	0 0 1 0
D	F	EQ	EP	0 0 1 1
T_1	L	EM	EH	0 1 0 0
X	Z	ER	EU	0 1 0 1
O	N	ET_2	EK	0 1 1 0
W	Y	EV	ES	0 1 1 1
EI	EC	G	Ø	1 0 0 0
EA_2	EA_1	B	π	1 0 0 1
EC_1	EC_2	T	J	1 0 1 0
ED	EF	Q	P	1 0 1 1
ET_1	EL	M	H	1 1 0 0
EX	EZ	R	U	1 1 0 1
EO	EN	T_2	K	1 1 1 0
EW	EY	V	S	1 1 1 1

Note: Each operator appears only once and there is another operation which produces a code which is the complement to the code produced by the given operation. There are two operations that produce codes totally orthogonal to the code produced by the first operation and its complement.

demonstrated that the 64 transformations are partitioned into 16 subgroups of code, complement, orthogonal code, and complement to the orthogonal code. See Table III.

3. Linear Homogeneous Equation Approach

Another way of expressing correlation is by a matrix multiplication as shown below:

$$\text{Let}\quad C = c_1, c_2, \cdots, c_n$$

$$A = a_1, a_2, \cdots, a_n$$

$$F = f_1, f_2, \cdots, f_{2n-1}$$

$$C \otimes A = F$$

This correlation can be represented as shown below:

$$(c_1, c_2, \cdots, c_n) \begin{bmatrix} a_n & a_{n-1} \cdots a_1 & 0 \cdots 0 \\ 0 & a_n \; a_{n-1} \quad a_1 & 0 \\ 0 \cdots 0 & a_n \; a_{n-1} \cdots & a_1 \end{bmatrix} = (f_1, f_2, \cdots, f_{2n-1})$$

$$C \cdot A_m = F$$

The subscript m is used to denote a matrix which is more than just a row matrix. This notation is used for convenience to show the form of the matrix equations.

For complementary correlation, let the n-dimensional vectors C and D represent one code and the n-dimensional vectors A and B represent the second code. Let channel one correlation be the vector F of dimension 2n-1 and channel two correlation similarly be the vector G. Let the vector H be the complementary correlation or the sum of F and G.

$$(c_1,\cdots,c_n,d_1,\cdots,d_n)\begin{bmatrix} a_n & \cdot & \cdot & a_2 & a_1 & 0 & 0 & 0 & 0 \\ 0 & a_n & \cdot & \cdot & a_2 & a_1 & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & a_n & \cdot & \cdot & a_2 & a_1 \\ b_n & \cdot & \cdot & b_2 & b_1 & 0 & 0 & 0 & 0 \\ 0 & b_n & \cdot & \cdot & b_2 & b_1 & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & b_n & \cdot & \cdot & b_2 & b_1 \end{bmatrix} = (f_1 + g_1,\cdots,f_{2n-1} + g_{2n-1})$$

$$= (h_1,\cdots,h_{2n-1})$$

Note that this matrix has 2n rows and 2n-1 columns.

For convenience the transpose of both sides is taken.

$$\begin{bmatrix} a_n & 0 & 0 & 0 & 0 & b_n & 0 & 0 & 0 & 0 \\ \cdot & a_n & 0 & 0 & 0 & \cdot & b_n & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ a_2 & \cdot & \cdot & \cdot & 0 & b_2 & \cdot & \cdot & \cdot & 0 \\ a_1 & a_2 & \cdot & \cdot & a_n & b_1 & b_2 & \cdot & \cdot & b_n \\ 0 & a_1 & a_2 & \cdot & \cdot & 0 & b_1 & b_2 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & a_2 & 0 & 0 & 0 & \cdot & b_2 \\ 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & b_1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ c_n \\ d_1 \\ d_2 \\ \cdot \\ \cdot \\ d_n \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \cdot \\ \cdot \\ \cdot \\ h_{2n-1} \end{bmatrix}$$

To find the totally orthogonal complementary pair involves solving the linear homogeneous equation with 2n unknowns and 2n-1 equations. By homogeneous is meant that the H vector is the null vector. This can be represented by the matrix equation:

$$A_m X = 0$$

A_m is the matrix with 2n-1 rows and 2n columns. The answer to this problem has been worked in many books [Ref. 4].

$$x_i = k \mid A_{m_i} \mid$$

The matrix A_{m_i} is $(-1)^{(i+1)} A_m$ matrix with the i_{th} column crossed out, and k is an arbitrary constant.

a. Example:

Using the linear homogeneous equation approach, find the complementary pair totally orthogonal to the complementary pair, A = (1,1), B = (1,-1).

$$\begin{bmatrix} +1 & 0 & -1 & 0 \\ +1 & +1 & +1 & -1 \\ 0 & +1 & 0 & +1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ d_1 \\ d_2 \end{bmatrix} = k \begin{bmatrix} \begin{vmatrix} 0 & -1 & 0 \\ +1 & +1 & -1 \\ +1 & 0 & +1 \end{vmatrix} \\ \begin{vmatrix} +1 & -1 & 0 \\ +1 & +1 & -1 \\ 0 & 0 & +1 \end{vmatrix} \\ \begin{vmatrix} +1 & 0 & 0 \\ +1 & +1 & -1 \\ 0 & +1 & +1 \end{vmatrix} \\ \begin{vmatrix} +1 & 0 & -1 \\ +1 & +1 & +1 \\ 0 & +1 & 0 \end{vmatrix} \end{bmatrix} = k \begin{bmatrix} 2 \\ -2 \\ 2 \\ 2 \end{bmatrix}$$

Restricting c_1, c_2, d_1, and d_2 to +1, and -1 the possible solutions are:

First solution $(c_1\ c_2\ d_1\ d_2) = (+1\ -1\ +1\ +1)$

Second solution $(c_1\ c_2\ d_1\ d_2) = (-1\ +1\ -1\ -1)$

Clearly the linear homogeneous equation can become very difficult to solve because of the size of the determinants even for comparably short codes. For the case where n=4 the solution involves solving eight seven dimensional determinants. However no matter how we get a solution, a constant times the solution is also a solution. Thus restricting ourselves to +1 and -1's the solution found previously (the reverse, exchange, and the complement of one) and its negative are both solutions.

The question arises as to whether it is possible that there could be another solution orthogonal to both the original complementary pair and to the solution that was found.

b. Theorem:

There is no complementary code totally orthogonal to both the original code and to one orthogonal code already found by reversing, exchanging, and complementing either one.

The correlation of both the original code and the orthogonal code found with a new code can be hypothesized. Let the solution be $(c_1,\cdots,c_n,d_1,\cdots,d_n)$. Then being orthogonal to the first involves 2n-1 equations and being orthogonal to the second involves another 2n-1 equations. These can be gathered together into one matrix.

$$
\begin{bmatrix}
a_n & 0 & 0 & 0 & 0 & b_n & 0 & 0 & 0 & 0 \\
\cdot & a_n & 0 & 0 & 0 & \cdot & b_n & 0 & 0 & 0 \\
\cdot & \cdot & a_n & 0 & 0 & \cdot & \cdot & b_n & 0 & 0 \\
a_2 & \cdot & \cdot & a_n & 0 & b_2 & \cdot & \cdot & b_n & 0 \\
a_1 & a_2 & \cdot & \cdot & a_n & b_1 & b_2 & \cdot & \cdot & b_n \\
0 & a_1 & a_2 & \cdot & \cdot & 0 & b_1 & b_2 & \cdot & \cdot \\
0 & 0 & a_1 & a_2 & \cdot & 0 & 0 & b_1 & b_2 & \cdot \\
0 & 0 & 0 & a_1 & a_2 & 0 & 0 & 0 & b_1 & b_2 \\
0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & b_1 \\
\bar{b}_1 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 \\
\bar{b}_2 & \bar{b}_1 & 0 & 0 & 0 & a_2 & a_1 & 0 & 0 & 0 \\
\cdot & \bar{b}_2 & \bar{b}_1 & 0 & 0 & \cdot & a_2 & a_1 & 0 & 0 \\
\cdot & \cdot & \bar{b}_2 & \bar{b}_1 & 0 & \cdot & \cdot & a_2 & a_1 & 0 \\
\bar{b}_n & \cdot & \cdot & \bar{b}_2 & \bar{b}_1 & a_n & \cdot & \cdot & a_2 & a_1 \\
0 & \bar{b}_n & \cdot & \cdot & \bar{b}_2 & 0 & a_n & \cdot & \cdot & a_2 \\
0 & 0 & \bar{b}_n & \cdot & \cdot & 0 & 0 & a_n & \cdot & \cdot \\
0 & 0 & 0 & \bar{b}_n & \cdot & 0 & 0 & 0 & a_n & \cdot
\end{bmatrix}
\begin{bmatrix}
c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \\ d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_n
\end{bmatrix}
=
\begin{bmatrix}
0 \\ \cdot \\ \cdot \\ \cdot \\ 0
\end{bmatrix}
$$

This is in the form of $A_m X = 0$. In this case there are 2n unknowns and 2(2n-1) equations. A_m has rank less than or equal to 2n, because it only has 2n columns. It remains to show A_m has rank 2n.

From linear algebra the rank of the product of matrices is not greater than any of its factors [Ref. 5]. For example:

$$A_m \cdot B_m = C_m$$

$$\text{rank}(C_m) \leq \min\ (\text{rank}(A_m),\ \text{rank}(B_m))$$

Thus a corollary of this is that the rank of the product of a matrix and its transpose is not greater than that of the matrix itself.

$$\operatorname{rank}(A_m^T \cdot A_m) \leq \operatorname{rank}(A_m)$$

$$A_m^T \cdot A_m = \begin{bmatrix} a_n & \cdots & a_1 & 0 \cdots 0 & \bar{b}_1 & \cdots & \bar{b}_n & 0 \cdots 0 \\ 0 & \ddots & & \ddots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots & & \ddots & 0 \\ 0 & \cdots 0 & a_n & \cdots & a_1 \, 0 & \cdots 0 & \bar{b}_1 & \cdots \, \bar{b}_n \\ b_n & \cdots & b_1 & 0 \cdots 0 & a_1 & \cdots & a_n & 0 \cdots 0 \\ 0 & \ddots & & \ddots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots & & \ddots & 0 \\ 0 & \cdots 0 & b_n & \cdots & b_1 \, 0 & \cdots 0 & a_1 & \cdots \, a_n \end{bmatrix} \begin{bmatrix} a_n & 0 \cdots 0 & b_n & 0 \cdots 0 \\ \vdots & \ddots \ 0 & \vdots & \ddots \ 0 \\ a_1 & \cdots \ a_n & b_1 & \cdots \ b_n \\ 0 & \ddots \ 0 & \ddots & \vdots \\ 0 \cdots 0 & a_1 & 0 \cdots 0 & b_1 \\ \bar{b}_1 & 0 \cdots 0 & a_1 & 0 \cdots 0 \\ \vdots & \ddots \ 0 & \vdots & \ddots \ 0 \\ \bar{b}_n & \cdots \ \bar{b}_1 & a_n & \cdots \ a_1 \\ 0 & \ddots \ 0 & \ddots & \vdots \\ 0 \cdots 0 & \bar{b}_n & 0 \cdots 0 & a_n \end{bmatrix}$$

When multiplication is done to get the main diagonal terms of the product it is just the autocorrelation of a complementary pair sequence. Any other position is either a shifted correlation of a complementary pair or correlation with the reversed, exchange, complement of one which is totally orthogonal. In either of the last two cases the result is zero, but the main diagonal is 2n in each position.

Therefore:

$$A_m^T \cdot A_m = 2n \; I_{2n}$$

The rank of $A_m^T \cdot A_m$ is 2n. The rank of A_m is greater or equal to 2n, but we previously showed that it was less or equal to 2n. Therefore the rank of A_m is 2n.

Since the rank of A_m is 2n, there is no non-trivial solution to the equation, and no code totally orthogonal to both the original code and one of the totally orthogonal codes can exist.

Q. E. D.

III. PROPOSED COMMUNICATIONS SYSTEMS

A matched filter for a given signal can be built either by analog or digital means to a complementary coded signal. The signal itself and its complement can be used as the two signals mark and space, or one and zero. A number of variations are proposed, evaluated for different applications. The first category is variable length coding system dependent upon noise, and the second system proposed is a multiplexing system based on complementary coding.

A. VARIABLE LENGTH CODING

1. Two Length Codes for Same Filter

Dr. S. Jauregui, Jr., mentioned the possibility of having a communication system based upon using a long complementary pair filter for high noise levels and when noise levels were lower a shorter pulse sequence could be used, but the filter would remain the same.

Example:

$$\text{Let} \quad A = +1 \ +1$$

$$B = +1 \ -1$$

$$AB = +1 \ +1 \ +1 \ -1$$

$$A\bar{B} = +1 \ +1 \ -1 \ +1$$

Time reversing the bottom code,

$$+1 \ +1 \ +1 \ -1$$

$$+1 \ -1 \ +1 \ +1$$

This is a standard technique for generating longer codes, and the time reversing of the second code is a standard transformation, so this is a complementary sequence.

		Autocorrelation
+1 +1 +1 -1	+1 +1 +1 -1	-1 0 1 4 1 0-1
+1 -1 +1 +1	+1 -1 +1 +1	1 0-1 4-1 0 1
Complementary	correlation	0 0 0 8 0 0 0

A shorter sequence in the same filter is:

+1 +1	+1 +1 +1 -1	-1 0 2 2 1
+1 -1	+1 -1 +1 +1	1 0-2 2-1
Complementary	correlation	0 0 0 4 0

Dr. Jauregui demonstrated the same phenomenon by hand for length of 8. Investigation of the following codes was conducted using the IBM 360:

Length 16 with code lengths 8,4,2

Length 40 with code lengths 20,10

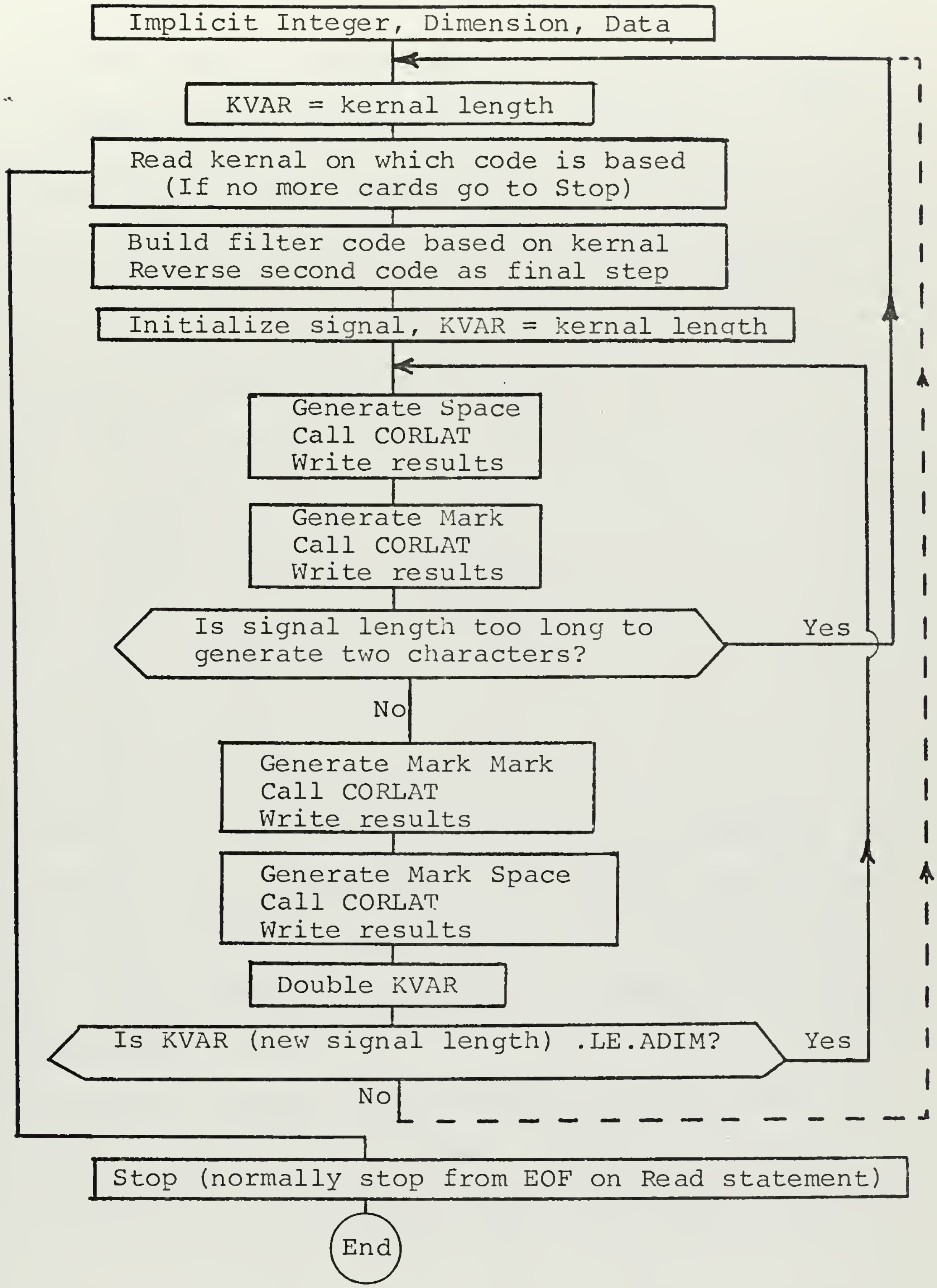
Length 104 with code lengths 52,26

The method of generating the codes is:

$$\begin{matrix} A_1 \\ B_1 \end{matrix} \longrightarrow \begin{matrix} A_1B_1 = A_2 \\ A_1\overline{B}_1 = B_2 \end{matrix} \longrightarrow \begin{matrix} A_2B_2 = A_3 \\ A_2\overline{B}_2 = B_3 \end{matrix} , \quad \text{etc.,}$$

where $\overline{B}$ is the complement of B. At the final step the lower code is time reversed.

The digital computer flow chart is in Fig. 6. The results are summarized below. In each case n is the signal code length.



*Dashed line is not normally used. Exit is normally made from the middle loop.

Figure 6. Variable Length Pulse Digital Computer Flow Chart

Half filter code length correlation

Mark $0_1, \cdots, 0_{n-1}, n, 0_{n+1}, \cdots, 0_{3n-1}$

Space $0_1, \cdots, 0_{n-1}, -n, 0_{n-1}, \cdots, 0_{3n-1}$

Quarter filter length correlation

Mark $0_1, \cdots, 0_{n-1}, +n, 0_{n+1}, \cdots, 0_{2n-1}, +n, 0_{2n+1}, \cdots, 0_{5n-1}$

Space $0_1, \cdots, 0_{n-1}, -n, 0_{n+1}, \cdots, 0_{2n-1}, -n, 0_{2n+1}, \cdots, 0_{5n-1}$

One-eighth filter length correlation

All terms are zero except as noted below:

Mark		Space
Term	Value	Value
n	-n	+n
2n	+n	-n
3n	+n	-n
4n	+n	-n

For codes of length one-half of the filter length, a single output pulse is generated. For shorter length codes the multiple pulse output makes the decoding somewhat ambiguous if the signals are sent sequentially immediately following the previous one.

An explanation of why this phenomenon works so well for codes of half length is given below. Taking the final two steps of building the code (underlining means reversing):

Given

$$\begin{bmatrix} A \\ B \end{bmatrix} \xrightarrow[\text{a new code}]{\text{Generating}} \begin{bmatrix} A & B \\ A & \bar{B} \end{bmatrix} \xrightarrow[\text{second code}]{\text{Reversing}} \begin{bmatrix} A & B \\ \underline{A\ \bar{B}} \end{bmatrix} \xrightarrow{\text{Simplifying}} \begin{bmatrix} A & B \\ \underline{\bar{B}} & \underline{A} \end{bmatrix}$$

The first part of the code is totally orthogonal to the second part. This can be demonstrated by showing that the second part is the exchange, reverse, complement of one transformation.

$$\begin{bmatrix} A \\ \overline{B} \end{bmatrix} \xrightarrow{\text{Exchange}} \begin{bmatrix} \overline{B} \\ A \end{bmatrix} \xrightarrow{\text{Time inverse}} \begin{bmatrix} \overline{B} \\ \underline{A} \end{bmatrix} \xrightarrow{\text{Complement one}} \begin{bmatrix} B \\ \underline{A} \end{bmatrix}$$

Thus the first part of the code is orthogonal to the second part. Consider the code $\binom{C}{D}$. Let $\binom{C'}{D'}$ be its totally orthogonal code. Since correlation is linear the following can be written:

$$\begin{bmatrix} C \\ D \end{bmatrix} \otimes_c \begin{bmatrix} C & C' \\ D & D' \end{bmatrix} = \begin{bmatrix} C \\ D \end{bmatrix} \otimes_c \begin{bmatrix} C & 0 \\ D & 0 \end{bmatrix} + \begin{bmatrix} C \\ D \end{bmatrix} \otimes_c \begin{bmatrix} 0 & C' \\ 0 & D' \end{bmatrix}$$

Thus:

$$\begin{bmatrix} C \\ D \end{bmatrix} \otimes_c \begin{bmatrix} C & C' \\ D & D' \end{bmatrix} = \begin{bmatrix} 0_1, \cdots, 0_{n-1}, n, 0_{n+1}, \cdots, 0_{3n+1} \end{bmatrix}$$

2. Totally Orthogonal Code Scheme

By alternately sending the coded signal and the orthogonal coded signal, it is possible to achieve maximum time separation of the different information signal on the two channels. See Fig. 5 for equipment setup.

The advantage to this system is that if multiple paths tend to spread signals over a period of time, the orthogonal sets keep the channels separated in time as far as possible. For instance in the above case with code lengths of four, the first message bit is on the standard code. The next bit on this code would come 8 bits later. In the meantime a message is decoded on the orthogonal code.

The total number of bits of information is as if the length of the code was four bits but the separation time on each channel is the time to send 8 bits. In the intervening time all signal generated clutter is canceled between the two channels. In other words, the transmitter can broadcast continuously, but when each message is reconstructed at the received, it is like an impulse without any side lobes which in a scatter environment could be falsely interpreted as signals.

B. MULTIPLEXING

The property that eliminates clutter from the radar system allows the improvement in resolution for a series of pulses is the orthogonality of the code with a time shift of itself. In a communication system this would allow the sender to start the next message one bit behind the previous one and by superposition of this linear system, it can be seen that these bits will be separated at the receiver end.

For simplicity the synchronous analog ideal receiver will be assumed for the uncoded pulse. The white gaussian noise will cause gaussian noise at all frequencies. Therefore it will have gaussian noise out of the synchronous detector.

The signal may be put on a carrier, but at the receiver a synchronous receiver is hypothesized. For the matched filter, the impulse response is just the time reverse of the signal. Then the normal convolution of the signal and filter becomes a correlation of the signal with itself. For convenience a square pulse is assumed for the uncoded system.

One way to get this impulse response is to use an integrator. The two signals are two pulses of equal magnitude and length but opposite phase. The initial condition of the integrator is zero. Integration proceeds during the length of the impulse. At the end of the baud, the value is sampled. The integrator is then reset for the next pulse. If it is positive the decision is made that the pulse was in phase, if negative the decision is made that the pulse is out of phase.

For a longer pulse, say n times the original length, either the signal could be sampled n times, the values added together and the decision made on the sum or the integrator could be sampled at the end of the total length.

For a coded pulse the signal can be sampled at each part and by multiplying either by plus or minus one according to the code and summing a decision can be made at the end as to which signal was sent.

Similarly for the complementary code, the scheme for a coded pulse is applied to both channels and the results are added together. The decision is made based on the sum of both channels.

Since the output of the matched filter itself is gaussian, it can be simulated by a gaussian noise generator. Since the uncoded matched filter is a part of the complementary code matched filter simulation, this can also be done for the multiplexed system, and thus provides a valid comparison, between the basic uncoded system, the basic complementary system, the multiplexed system using a single code, and the full

multiplexed system. The signal to noise ratio is the power ratio measured at the output of the basic matched filter system.

For the uncoded system the noise is assumed gaussian at the output of the matched filter. The probability of an error is:

P(error) = P(mark)·P(error mark) + P(space)·P(error space).

The probability of error given a mark is the same as the probability of an error given a space, and the probability of a mark plus the probability of a space is one. Then:

$$P(\text{error}) = 1 - F(x)$$

Where F(x) is the cumulative distribution function of the normal or gaussian probability, function x is the square root of the signal to noise power ratio.

A complementary code of length sixteen was used to encode a message of length one thousand bits and the number of errors were recorded. The signal to noise ratio required for any error percentage was 32 times lower, but two channels were needed and the flow of information was 16 times slower (See Fig. 7).

Full multiplexing using both the standard and totally orthogonal codes was conducted for codes of length 128, 64, 32, 16, 8, 4, and 2. All produced results in close agreement with the basic uncoded and the theoretical curve based on gaussian distribution for the uncoded system (See Fig. 8).

If higher data rate is desired the series of pulses in the code can be compressed into a shorter time. This results

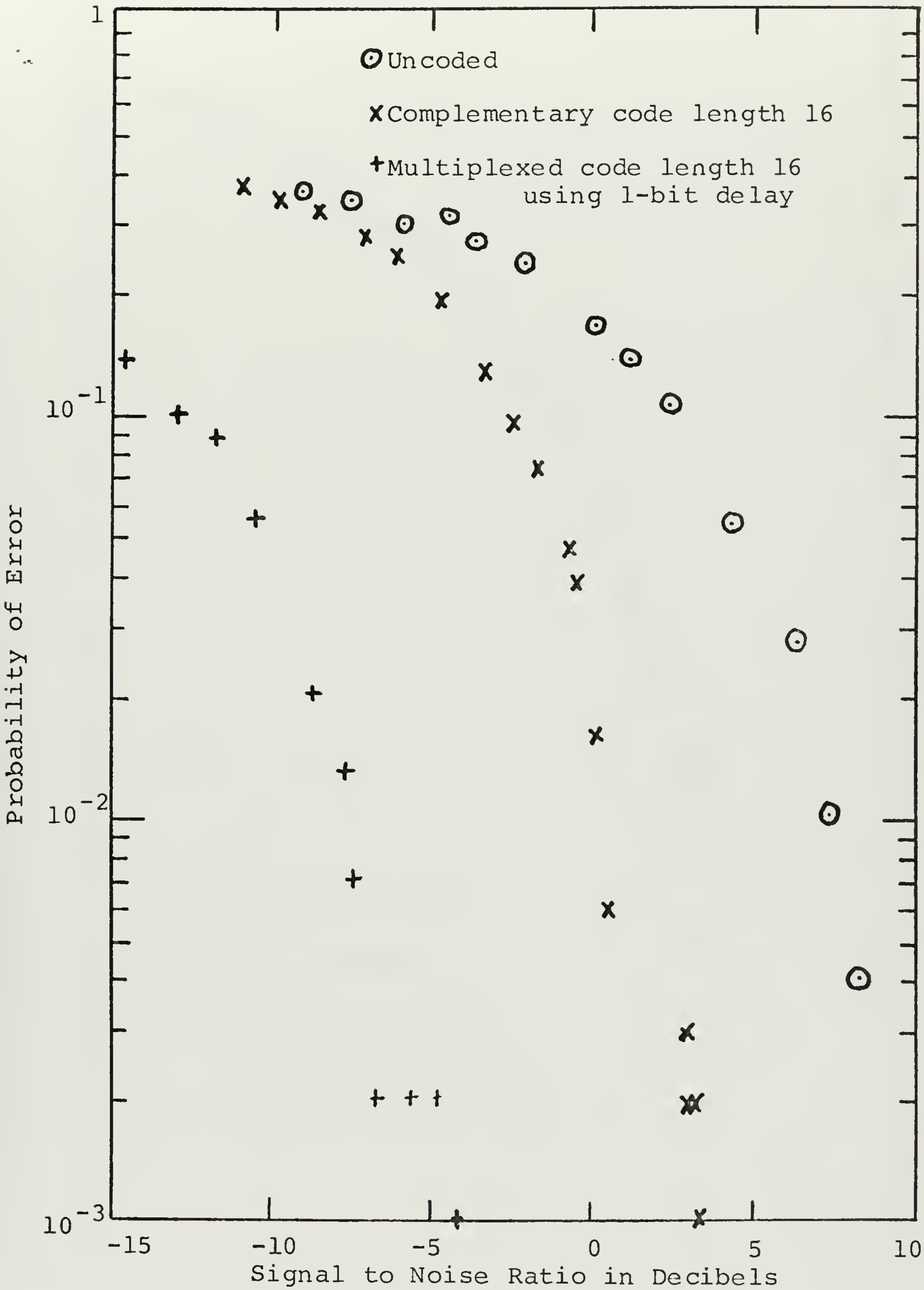


Figure 7. Probability of Error Versus Signal to Noise Ratio for Various Coded Signals.

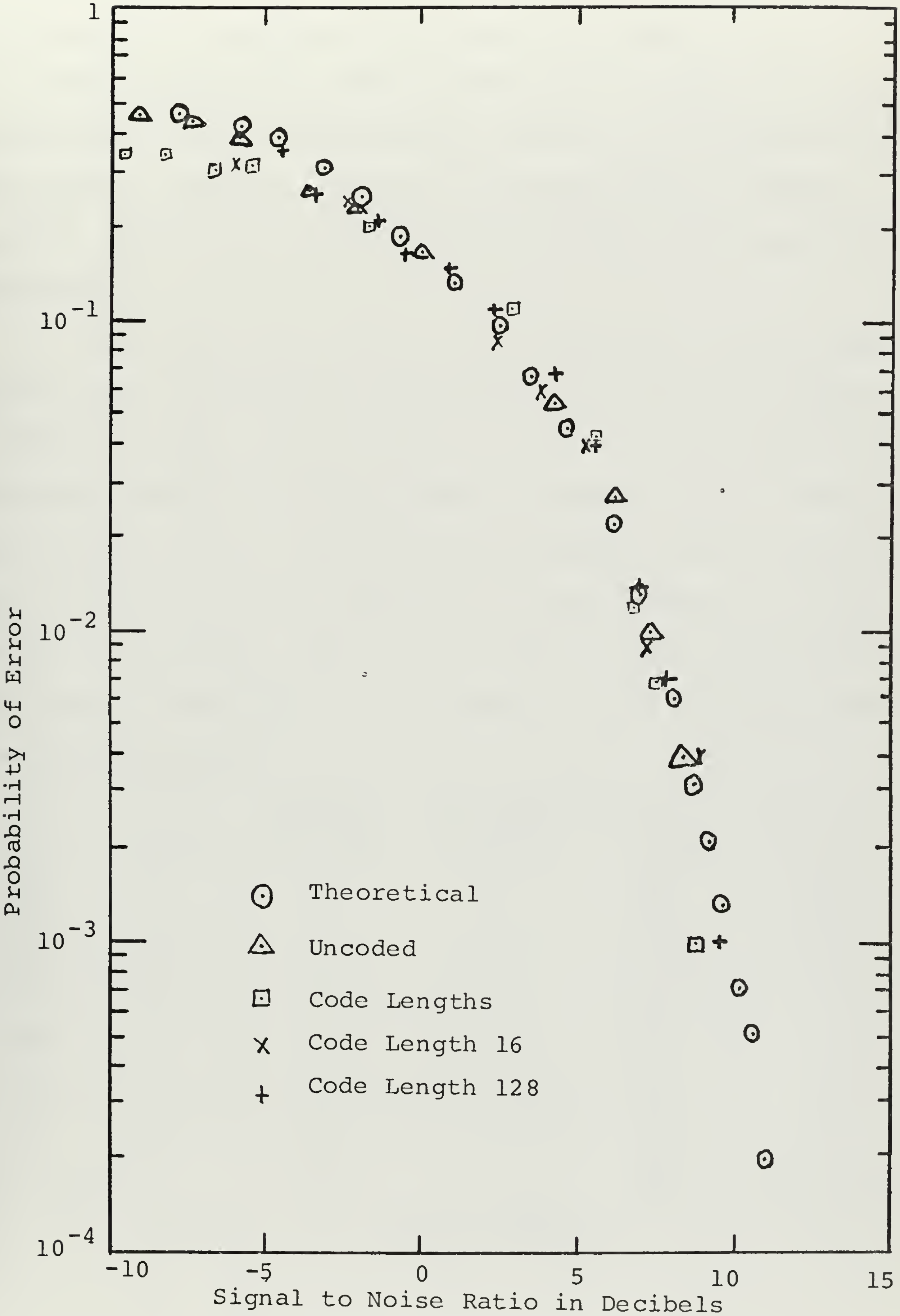


Figure 8. Probability of Error Versus Signal to Noise Ratio for Fully Multiplexed Systems with Various Length Codes

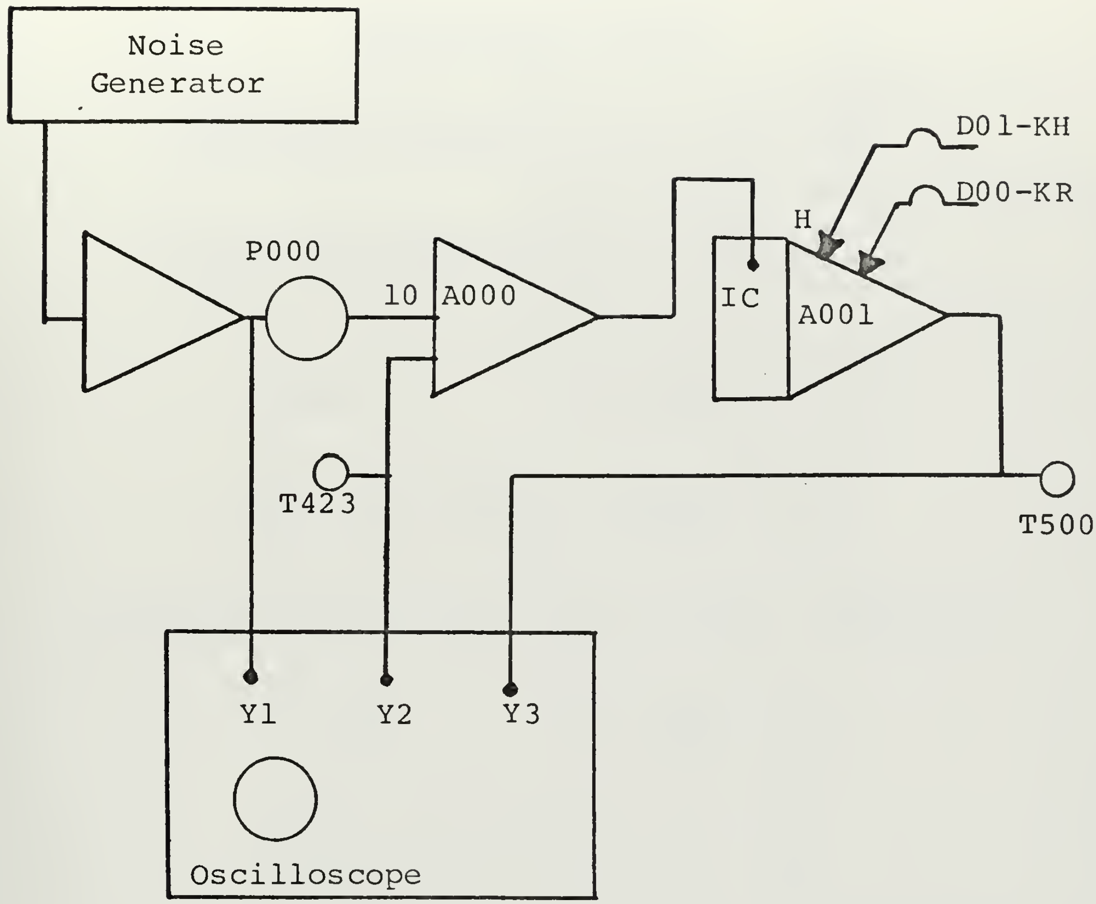
in a wider bandwidth, but using the horizontal multiplexing technique, the bandwidth can be fully utilized.

Depending upon the noise background the amount of signals multiplexed onto a complementary channel can be limited to get any desired accuracy. If very low noise level is present another method may be used to again increase the transmitted information.

An additional fifty percent of information can be transmitted over the bandwidth and channel if the noise level is sufficiently low. Three bits are taken at a time either from three sources or from one source at a higher rate. This is converted to a base three number and mapped into the plus one, minus one, or zero and encoded. At the receiver it is received as plus one, minus one, or zero on both standard and the totally orthogonal filter, mapped into a ternary system converted to binary.

3 bits	Ternary	Standard Filter	Orthogonal Filter	Ternary	Decoded
000	00	-1	-1	00	000
001	01	-1	1	01	001
010	02	-1	0	02	010
011	10	1	-1	10	011
100	11	1	1	11	100
101	12	1	0	12	101
110	20	0	-1	20	110
111	21	0	1	21	111
No signal	22	0	0	22	No signal

Mapping		Reverse Mapping	
0	-1	+1	1
1	1	0	2
2	0	-1	0



T-423 is a digital to analog trunk line. By calling the subroutine DAC a voltage specified from the digital computer is set on that trunk line. T-500 is an analog to digital trunk line. By calling the subroutine ADK the analog signal present at this trunk line is sampled, converted to digital, and stored in the digital computer under the variable specified.

Integrator A-001 is used as a track-hold network; holding the voltage while in compute mode, and tracking the voltage when the analog computer is in reset. The capacitor selected for A-001 was the smallest available, .001 microfarads to allow very rapid reset times.

The noise generator was fed into an amplifier to maintain isolation under all settings of P-000.

Figure 9. Analog Computer Diagram

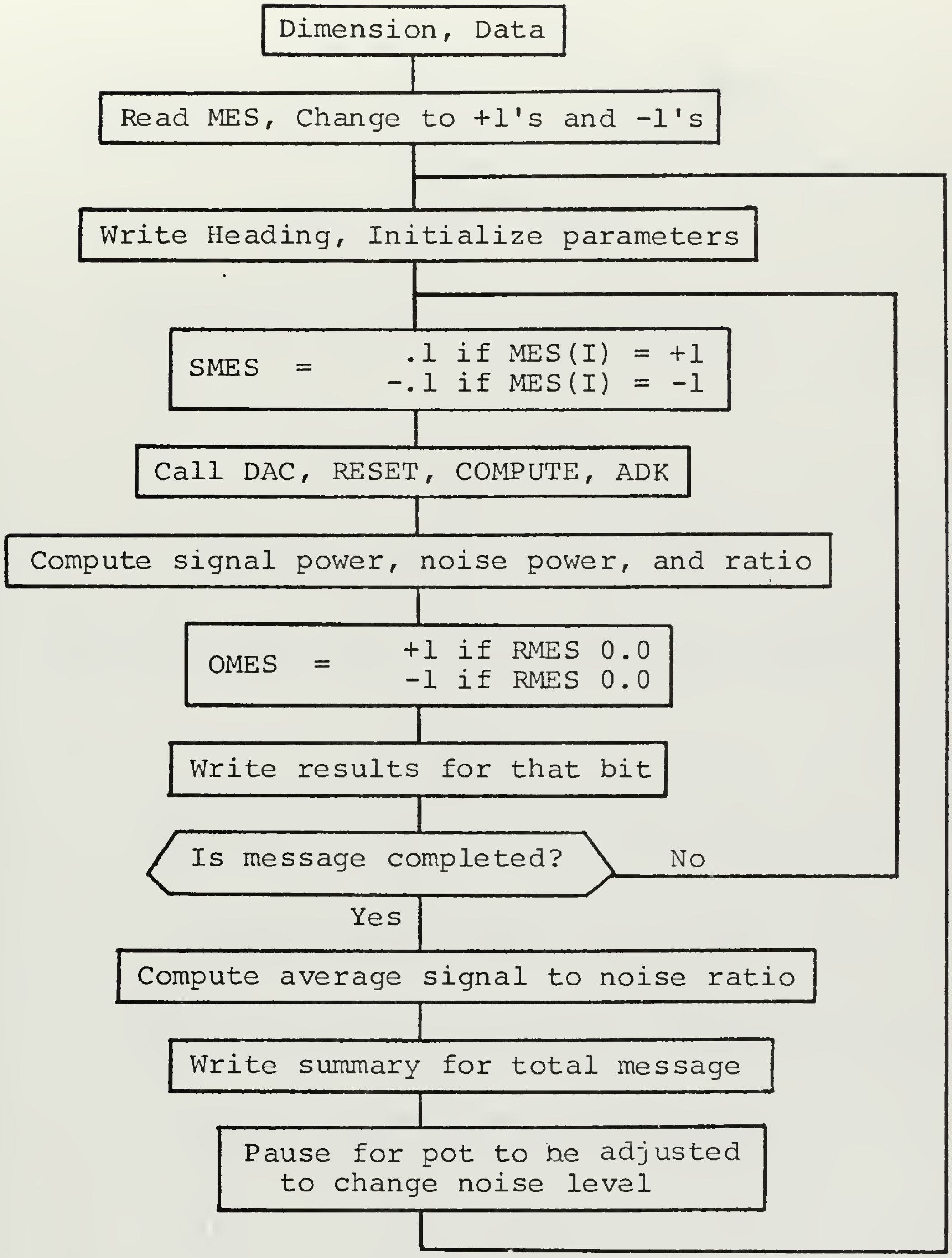


Figure 10. Digital Computer Flow Chart of Uncoded Pulses

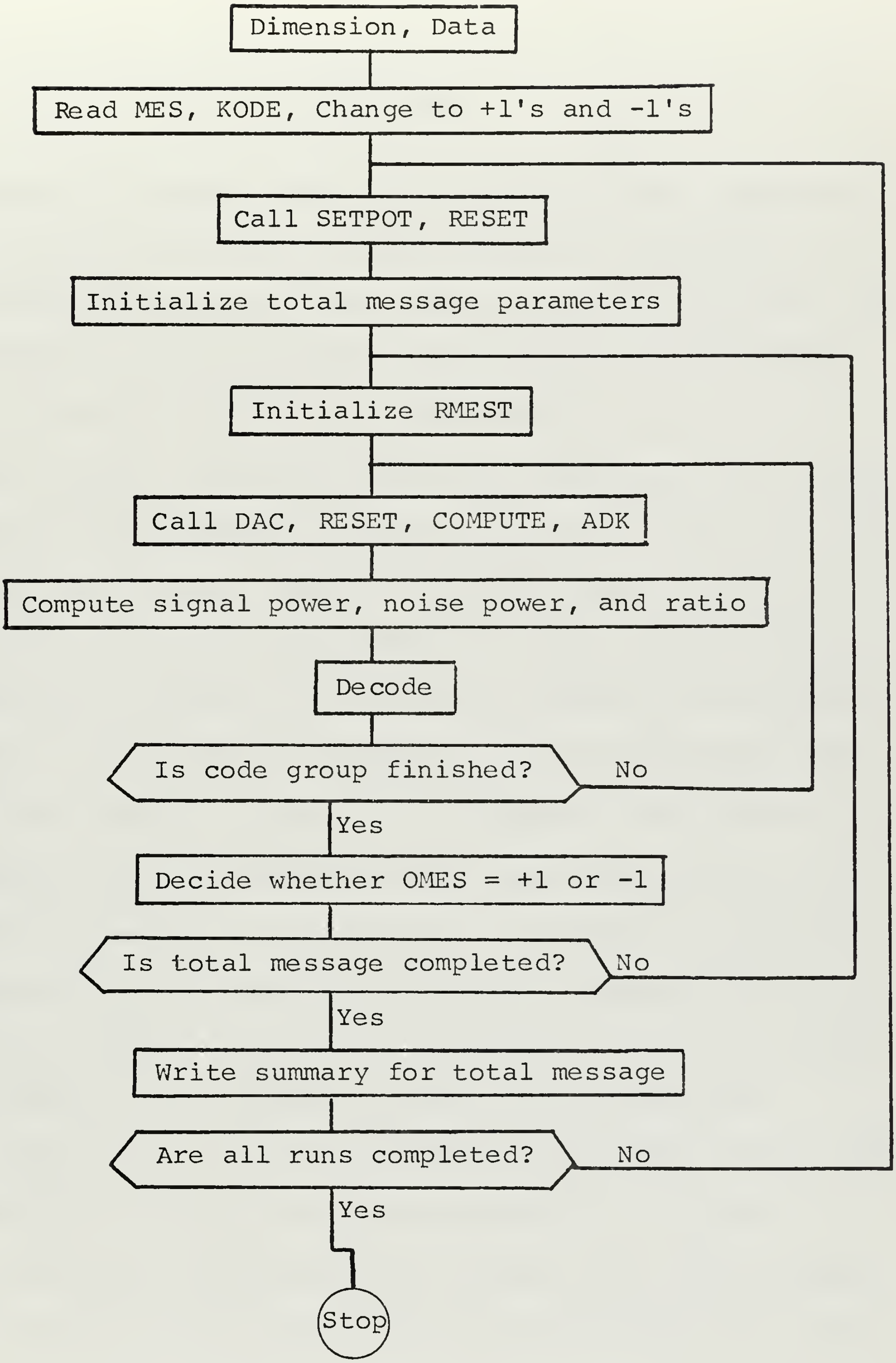


Figure 11. Digital Computer Flow Chart of Unmultiplexed Complementary Coded System

IV. CONCLUSION

1. Totally orthogonal complementary binary coded sequences were found to be the exchange, reverse, complement one transformation of the basic code. The basic code and its complement were both proven to be totally orthogonal to the orthogonal code and its complement.

2. These totally orthogonal codes were developed with an eye toward communications, however applications also exist in sonar, and possibly radar. If nothing else it would be possible to double the unambiguous range for any given repetition frequency by alternately sending the standard code and the totally orthogonal code and receiving both on the standard code and the orthogonal code filters.

3. Depending on the noise level the code length can be adjusted as necessary to allow the transmitter to send however much energy per bit necessary to communicate reliably. However the data rate has to go down during noisy times to maintain the necessary fidelity of information. Virtually any rate of multiplexing can be done on the same bandwidth to obtain the error probability required.

4. In the system simulated white gaussian noise was assumed, but various burst noise levels could be simulated, and if the duration of the burst were short, the signal can still be decoded correctly. Since the signals are really spread over a long time on the two channels by deleting a few

bits here and there would not appreciably affect the net result. This method could be very advantageous if used properly.

5. Advantages of this multiplex system would be the ease of splitting the signal up using digital techniques, and without the need for guard bands as in the frequency multiplex system. An advantage of this system over the time multiplex system is the bit error correction in a burst channel noise situation. Among the inefficiencies of needing several levels of signals in the output transmitter. The need for two channels might be overcome by sending the two signals in quadrature, or even time multiplexing if the signal is to be digitally processed.

APPENDIX A

SIMULATED COMMUNICATION SYSTEM PROGRAMS

Three digital programs are included in this thesis. The first two correspond to the flow charts in Figs. 10 and 11 for the uncoded system and simple coded systems. The third is the full multiplexed system using both the totally orthogonal code and the one bit delay.

All programs were debugged on short codes and the results of each bit was recorded, but after debugging only the summary of each run was printed. This was accomplished by using the X in the comment column of the write cards which were to be eliminated. By using the control card "FORTRAN LS,GO,X" these are compiled, but when "FORTRAN LS,GO" is used, these cards are treated as comments.

The length of code can be changed by changing input data and three cards: (1) DIMENSION statement, (2) DATA statement for KDIM, and (3) 8501 FORMAT statement. Similarly the message length can be changed by changing the (1) DIMENSION statement, (2) DATA statement for MESDIM, and (3) 8500 FORMAT statement.

On the third program it was necessary to change some simple variables into arrays. The message of the full multiplexed system requires one bit less than the sum of the message length and the code length. On fairly short messages and long codes this could distort the amount of noise power

in the calculation. This program corrected for this to make the proportion the same as for the case that the message is much longer than the code length.

PRINCIPAL VARIABLES USED

CODE	complementary code(s) in real numbers.
DBSN	signal to noise level in decibels.
IERR	increased by one if there is an error and is zero otherwise.
KDIM	dimension of complementary code.
KODE	complementary code array in integers.
KPOT	array of pot settings for various runs.
KPOTDM	dimension of KPOT.
MES	message composed of 0's and 1's read in for convenience and then changed to -1's and +1's.
MESDIM	length of the message
NERR	number of errors.
NUMBIT	used in totally orthogonal multiplexing due to two channels of message and is equal to 2 times MESDIM.
OMES	plus or minus one based on received signal.
QMES	array of preceding message bits, used in multiplex encoding.
RMES	received signal.
RMEST	sum of received coded bits, on which the final decision of whether the reception is +1 or -1.
SIGXN	signal to noise ratio of each bit.
SIGXNT	signal to noise ratio of total received message.
SMES	message sent (real numbers).
SPWR	signal power each bit (normalized to 1 ohm).
SPWRT	signal power in the entire message.
XNOISE	the difference between SMES and RMES, the noise voltage for each bit.

XPWR	noise power, i.e., XNOISE squared.
XPWRT	noise power for the total message.

```
ΔFORTRAN LS,GO
C       PROGRAM TO CHECK BUT BASIC UNCODED COMMUNICATION SYSTEM
        DIMENSION MES(72)
        DATA MESDIM/72/
        READ (5,8500)MES
        DO 100 I=1,MESDIM
        MES(I)=2*MES(I)-1
    100 CONTINUE
        WRITE (6,8900)
        SPWRT=0.0
        XPWRT=0.0
        NERR=0
        DO 200 I=1,MESDIM
        DATA IERR,OMES,SMES/0,1,0.1/
        SMES=0.1
        IF(MES(I).LT.0) SMES=-.1
        CALL DAC(4,SMES)
        CALL RESET(20)
        CALL COMPUTE
        CALL ADK(0,RMES)
C       RMES IS SIGNAL PLUS NOISE
        XNOISE=RMES-SMES
        XPWR=XNOISE**2
        SPWR=SMES**2
        SIGXN=SPWR/XPWR
        XPWRT=XPWRT+XPWR
        SPWRT=SPWR+SPWRT
        OMES=-1
         IF (RMES.GE.0.0) OMES=1
        IERR=0
        IF(MES(1).NE.OMES)IERR=1
        NERR=NERR+IERR
        WRITE(6,9000) MES(I),OMES,SIGXN
        IF (IERR.EQ.O) GO TO 200
        WRITE(6,9001)NERR
```

```
  200 CONTINUE
      SIGNT=SPWRT/XPWRT
      WRITE (6,9050)NERR,MESDIM,SIGNT
      INPUT(101)
      GO TO 100
 8500 FORMAT(72I1)
 8900 FORMAT('1 MESSAGE SENT'3X'MESSAGE RECEIVED '3X'SIGNAL TO NOISE RATI
     1O'///)
 9000 FORMAT(6X,1I2,16X1I2,10X,F15.4)
 9001 FORMAT('+'70X'ERROR'6X'NUMBER OF ERRORS'I5)
 9050 FORMAT(//23X'SUMMARY'//' THERE WERE'I5' ERRORS IN THE MESSAGE OF L
     1ENGTH'I7/' THE AVERAGE SIGNAL TO NOISE RATIO WAS'F15.4)
      STOP
      END
```

```
ΔFORTRAN LS,GO
C     PROGRAM TO CHECK OUT COMPLEMENTARY PAIR CODE SEQUENCE
      DIMENSION MES(1008),KODE(16,2),KPOT(30)
      DATA KPOT/9999,9000,8000,7000,6000,5000,4000,3500,3000,2500,2000,1
     1700,1500,1300,1100,1000,900,800,700,600,500,400,300,250,220,200,18
     20,150,100,50/
      DATA MESDIM/1008/
      DATA KDIM/16/
      READ (5,8501)KODE
      READ (5,8500)MES
      DO 50 I=1,KDIM
      KODE(I,1)=KODE(I,1)*2-1
      KODE(I,2)=KODE(I,2)*2-1
   50 CONTINUE
      DO 100 I=1,MESDIM
      MES(I)=2*MES(I)-1
  100 CONTINUE
      DO 300 I2=1,30
      CALL SETPOT(4HP000,KPOT(I2))
X     WRITE (6,8900)
      NERR=0
      SPWRT=0.0
      XPWRT=0.0
      DO 200 I=1,MESDIM
      RMEST=0.0
      DO 150 J=1,KDIM
      DO 150 I1=1,2
      K=KDIM+1-J
      SMES=0.01
      IF((KODE(K,I1)*MES(I)).LT.0) SMES=-.01
      CALL RESET(1)
      CALL DAC(4,SMES)
      CALL RESET(2)
      CALL COMPUTE
      CALL ADK(0,RMES)
C     RMES IS SIGNAL PLUS NOISE
```

```
      XNOISE=RMES-SMES
      XPWR=XNOISE**2
      SPWR=SMES**2
      SIGXN=SPWR/SPWR
      XPWRT=XPWRT+XPWR
      SPWRT=SPWR+SPWRT
      IF (KODE(K,I1).LT.0) RMES=-RMES
      RMEST=RMEST+RMES
  150 CONTINUE
      SMES=-1
      IF(RMEST.GE.0.0) OMES=1
      IERR=0
      IF(MES(I).NE.OMES)IERR=1
      NERR=NERR+IERR
X     WRITE(6,9000) MES(I),OMES,SIGXN
      IF (IERR.EQ.0) GO TO 200
X     WRITE (6,9001)NERR
  200 CONTINUE
      SIGNT=SPWRT/XPWRT
      WRITE (6,9050)NERR,MESDIM,SIGNT
  300 CONTINUE
 8500 FORMAT(13(72I1/1,72I1)
 8501 FORMAT(32I1)
 8900 FORMAT('1 MESSAGE SENT'3X'MESSAGE RECEIVED'3X'SIGNAL TO NOISE RATI
     1O'///)
 9000 FORMAT(6X,1I2,16X1I2,10X,F15.4)
 9001 FORMAT('+'70X'ERROR'6X'NUMBER OF ERRORS'I5)
 9050 FORMAT(//23X'SUMMARY'//' THERE WERE'I5' ERRORS IN THE MESSAGE OF L
     1ENGTH'I7/' THE AVERAGE SIGNAL TO NOISE RATIO WAS'F15.4)
      STOP
      END
```

TOTALLY ORTHOGONAL COMPLEMENTARY PAIRS USING ONE BIT DELAY   DATE

```
ΔFORTRAN LS,GO                                                              RTM
C     THIS ALGORITHM PERMITS KODE IN EITHER ONES AND ZEROS OR PLUS AND
C     MINUS ONES
      DIMENSION MES(500,2),KODE(128,2,2),KPOT(30),CODE(128,2,2),QMES(128
     1,2),RMES(128,2),OMES(2),RMEST(2)
      DATA MESDIM,KDIM,KPOTDM/500,128,30/
      DATA KPOT/9999,9000,8000,7000,6000,5000,4000,3500,3000,2500,2000,1
     1700,1500,1300,1100,1000,900,800,700,600,500,400,300,250,220,200,18
     20,150,100,50/
      NUMBIT=MESDIM*2
      READ (5,8501)KODE
      READ (5,8500)MES
      DO 50 I2=1,2
      DO 50 I1=1,2
      DO 50 I=1,KDIM
      CODE(I,I1,I2)=1.0
      IF(KODE(I,I1,I2).LE.0) CODE(I,I1,I2)=-1.0
   50 CONTINUE
      DO 100 I1=1,2
      DO 100 I=1,MESDIM
      MES(I,I1)=2*MES(I,I1)=1
  100 CONTINUE
      DO 300 I2=1,KPOTDM
      CALL SETPOT(4HP000,KPOT(I2))
      CALL RESET(500)
X     WRITE (6,8900)
      NERR=0
      SPWRT=0.0
      XPWRT=0.0
      DO 130 I1=1,2
      DO 130 14=1,KDIM
      QMES(14,I1)=0.0
```

```
      RMES(I4,I1)=0.0
  130 CONTINUE
      LOOP=MESDIM+KDIM-1
      DO 200 I=1,LOOP
      RMEST(1)=0.0
      RMEST(2)=0.
      DO 140 I5=2,KDIM
      I7=KDIM+2-15
      I6=I7-1
      QMES(I7,1)=QMES(I6,1)
      QMES(I7,2)=QMES(I6,2)
  140 CONTINUE
      IF(I.GT.MESDIM) GO TO 141
      QMES(1,1)=0.005
      IF(MES(I,1).LE.0) QMES(1,1)=-.005
      QMES(1,2)=0.005
      IF(MES(I,2).LE.0) QMES(1,2)=-.005
      GO TO 142
  141 QMES(1,1)=0.0
      QMES(1,2)=0.0
  142 DO 150 I1=1,2
      SMES=0.0
      DO 145 I9=1,KDIM
      I10=KDIM-I9+1
      SMES=SMES+CODE(I10,I1,1)*QMES(I9,1)+CODE(I10,I1,2)*QMES(I9,2)
  145 CONTINUE
X     OUTPUT(101) SMES
      CALL DAC(4,SMES)
      CALL RESET(1)
      DO 148 I5=2,KDIM
      I7=KDIM+2-I5
      I6=I7-1
      RMES(I7,I1)=RMES(I6,I1)
```

```
      148 CONTINUE
C         RMES IS SIGNAL PLUS NOISE
          CALL COMPUTE
          CALL ADK(0,RMES(1,I1))
X         OUTPUT(101) RMES(1,I1)
          XNOISE=RMES(1,I1)-SMES
          XPWR=XNOISE**2
          SPWR=SMES**2
X         SIGXN=SPWR/SPWR
          XPWRT=XPWRT+XPWR
          SPWRT=SPWR+SPWRT
          DO 150 I4=1,KDIM
          RMEST(1)=RMEST(1)+RMES(I4,I1)*CODE(I4,I1,1)
          RMEST(2)=RMEST(2)+RMES(I4,I1)*CODE(I4,I1,2)
      150 CONTINUE
          IF(I.LT.KDIM)GO TO 200
X         OUTPUT(101)RMEST(1),RMEST(2)
          IERR=0
          I4=I-KDIM+1
          DO 170 I1=1,2
          OMES(I1)=-1
          IF(RMEST(I1).GE.0.0) OMES(I1)=1
          IF (MES(I4,I1).NE.OMES(I1))IERR=IERR+1
      170 CONTINUE
          NERR=NERR+IERR
X         WRITE(6,9000) MES(I4,1),MES(I4,2),OMES(1),OMES(2),SIGXN
          IF(IERR-1) 200,191,192
      191 CONTINUE
X         WRITE(6,9001)NERR
          GO TO 200
      192 CONTINUE
X         WRITE(6,9002) NERR
      200 CONTINUE
```

TOTALLY ORTHOGONAL COMPLEMENTARY PAIRS USING ONE BIT DELAY

```
      SIGNT=SPWRT/XPWRT
      SIGNT=SIGNT*FLOAT(MESDIM+KDIM-1)/FLOAT(MESDIM)
      DBSN=10.0*FLOAT(DLOG10(SIGNT))
      WRITE(6,9050)NERR,NUMBIT,SIGNT,DBSN
      IF(SENSE SWITCH 1) 350,300
  300 CONTINUE
      GO TO 9100
  350 IF(SENSE SWITCH 1) 350,100
 8500 FORMAT(13(72I1/),64I1)
 8501 FORMAT(31(16I1/),16I1)
 8900 FORMAT('1 MESSAGE SENT'3X'MESSAGE RECEIVED'3X'SIGNAL TO NOISE RATI
     1O'///)
 9000 FORMAT(4X,I2,2X,I2,12X,I2,2X,I2,6X,F15.4)
 9001 FORMAT('+'70X'ERROR'6X'NUMBER OF ERRORS'I5)
 9002 FORMAT('+'63X'DOUBLE ERROR'6X'NUMBER OF ERRORS'I5)
 9050 FORMAT(//23X'SUMMARY'//' THERE WERE'I5' ERRORS IN THE MESSAGE OF L
     1ENGTH'I7/' THE AVERAGE SIGNAL TO NOISE RATIO WAS'F15.4/' THE SIGNA
     2L TO NOISE RATIO IN DECIBELS IS'F8.3)
 9100 CONTINUE
      STOP
      END
```

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13. ABSTRACT

Complementary binary code sequences were invented by M. J. E. Golay in the investigation of infrared multislit spectrometry. Complementary coding sequences have the property of an infinite correlation peak to peak ambiguity ratio when detected with a matched filter.

Cooperative or totally orthogonal complementary code pairs are two sets of complementary pairs such that the cross correlation is zero in every position. A proof is given that every complementary pair has two totally orthogonal pairs, i.e., one the complement of the other. A proof that these pairs are the only pairs is also given.

A communication system involving complementary code binary sequences is simulated on the hybrid computer and compared with an ideal receiver for an uncoded signal. By using both the totally orthogonal code and time shifting, a method of horizontal multiplexing of binary coded information is proposed and evaluated. Various other complementary coding systems and possible uses are discussed.

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KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Complementary Sequence						
Complementary Series						
Complementary Codes						
Golay Codes						
Orthogonal Codes						
Communication Codes						
Compressive Codes						
Totally Orthogonal Codes						
Multiplex Communication System						

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Totally orthogonal complementary binary codes sequences and applications to communications systems.